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APPLICATIONS OF HYPERSEMIGROUPS TO INTERVAL-VALUED FUZZY SUBSETS

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Abstract. *The purposes of this paper are to introduce generalizations of interval-valued fuzzy hypersemigroups to the context of interval-valued fuzzy hypersemigroups. We also discuss some basic properties of interval-valued fuzzy hypersemigroups and characterize the interval-valued fuzzy hypersemigroups. Moreover, we define interval-valued fuzzy subhypersemigroups, interval-valued fuzzy hyperideal (right interval-valued fuzzy hyperideal, left fuzzy hyperideal), interval-valued fuzzy hyper bi-ideals and interval-valued fuzzy hyper interior ideals in interval-valued fuzzy hypersemigroups and some properties of them are obtained. At the end we characterize interval-valued fuzzy hyper interior ideals and interval-valued fuzzy hyper bi-ideals. In this regard, we prove that an interval-valued fuzzy subhypersemigroup* $\widetilde{\mu}$ *of an interval-valued fuzzy hypersemigroup* S *is an interval-valued fuzzy hyper interior ideal of* S *if and only if* $\widetilde{S} \circ \widetilde{\mu} \circ \widetilde{S} \subseteq \widetilde{\mu}$.
Keywords: Interval-valued fuzzy hypersemigroup, Interval-valued fuzzy hyper interior ideal, Interval-valued fuzzy hyperideal, Iinterval-valued fuzzy hypersimple, Interval-valued fuzzy hyper bi-ideal

1. **Introduction.** In 1975, Zadeh [15] introduced the notion of an interval-valued (i.v.) fuzzy subset. In 2007, Davvaz [3] studied some properties of (fuzzy) hyperideals in H_v-semigroups. In 2008, Sen et al. [12] introduced the concept of the fuzzy hypersemigroup as a generalization of semigroup and fuzzy subset. In 2009, Davvaz et al. [5] introduced and studied the notion of quasicoincidence in a fuzzy interval value with an interval-valued fuzzy set. Davvaz and Fotea [4] defined interval-valued (anti) fuzzy n-ary subpolygroups. In 2010, Kazanci et al. [8] introduced a new type of fuzzy n-ary sub-hypergroups in an n-ary hypergroup, that is, the $(\in, \in \vee_q)$-fuzzy n-ary sub-hypergroups. In 2015, Kar et al. [7] introduced and studied the notion of interval-valued fuzzy hyperideals of semihypergroups. In 2016, Nozari [10] has studied commutative fundamental relations in fuzzy hypersemigroups. Abdullah et al. [1] have studied interval valued intuitionistic fuzzy bi-Γ-hyperideals, interval valued intuitionistic fuzzy $(1, 2) - \Gamma$-hyperideals and interval valued intuitionistic fuzzy left (right, two sided) Γ-hyperideals of Γ-semihypergroups. Khan et al. [9] introduced the notion of $(\in, \in \vee_{q_k})$-cubic hyperideals, $(\in, \in \vee_{q_k})$-cubic bi-hyperideals, $(\in, \in \vee_{q_k})$-cubic generalized bi-hyperideals, $(\in, \in \vee_{q_k})$-cubic interior hyperideals and $(\in, \in \vee_{q_k})$-cubic quasi-hyperideals in LA-semihypergroups. In 2017, Ahmed et al. [2] introduced the concept of n-dimensional fuzzy sets, fuzzy hyperideals and fuzzy prime hyperideals in semihyperrings with identity. Kar and Purkait [6] defined interval-valued (i.v.) fuzzy k-quasi ideals and i.v. fuzzy k-bi-ideals of semihyperrings. In 2018, Sarkar and Kar [11] introduced and studied the notion of interval-valued (in short, (i-v)) prime fuzzy hyperideals in semihypergroups. In 2020, Yairayong [13] introduced

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and studied the fuzzy LA-subhypersemigroups, left fuzzy hyperideal (right fuzzy hyperideal, fuzzy hyperideal), left fuzzy hypersimples and fuzzy hyper bi-ideals of fuzzy LA-hypersemigroups and obtained its basic results. In [14], Yiarayong studied the notion of picture fuzzy subsemigroups.

In this paper, our aim is to introduce the concept of interval-valued fuzzy hypersemigroups. We also discuss some basic properties of interval-valued fuzzy hypersemigroups and characterize the interval-valued fuzzy hypersemigroups. Moreover, we define interval-valued fuzzy subhypersemigroups, left interval-valued fuzzy hyperideal (right interval-valued fuzzy hyperideal, interval-valued fuzzy hyperideal), interval-valued fuzzy hyper bi-ideals and interval-valued fuzzy hyper interior ideals in interval-valued fuzzy hypersemigroups and some properties of them are obtained. At the end we characterize interval-valued fuzzy hyper interior ideals and interval-valued fuzzy hyper bi-ideals.

2. **Interval-Valued Fuzzy Hypersemigroups.** In 2008, Sen et al. [12] introduced the concept of fuzzy hypersemigroups and studied its different properties. We need to extend the fuzzy hyperoperations to the interval-valued fuzzy hyperoperations.

Let S be a non empty set and $\widetilde{\mathcal{F}(S)}$ denotes the set of all interval-valued fuzzy subset of S. An **interval-valued fuzzy hyperoperation** on S is a mapping $\circ : S \times S \rightarrow \widetilde{\mathcal{F}(S)}$ written as $(x, y) \mapsto \widetilde{x} \circ \widetilde{y}$. A non empty S together with an interval-valued fuzzy hyperoperation "$\circ$" is called an **interval-valued fuzzy hypergroupoid**.

Based on [12], we can extend the concept of fuzzy hypersemigroups to the concept of interval-valued fuzzy hypersemigroups in the following way.

Definition 2.1. *An interval-valued fuzzy hypergroupoid $(S, \circ)$ is called an **interval-valued fuzzy hypersemigroup** if for all $x, y, z \in S$, $(\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z} = \widetilde{x} \circ (\widetilde{y} \circ \widetilde{z})$, where for any $\widetilde{\mu} \in \widetilde{\mathcal{F}(S)}$*

$$(\widetilde{x} \circ \widetilde{\mu})(a) = \begin{cases} \bigvee_{s \in S} (\widetilde{x} \circ \widetilde{s})(a) \wedge \widetilde{\mu}(s); & \text{if } \widetilde{\mu} \neq \widetilde{0} \\ \widetilde{0}; & \text{otherwise} \end{cases}$$

and

$$(\widetilde{\mu} \circ \widetilde{x})(a) = \begin{cases} \bigvee_{s \in S} \widetilde{\mu}(s) \wedge (\widetilde{s} \circ \widetilde{x})(a); & \text{if } \widetilde{\mu} \neq \widetilde{0} \\ \widetilde{0}; & \text{otherwise.} \end{cases}$$

The following four theorems provide us some examples of interval-valued fuzzy hypersemigroups.

Theorem 2.1. *Let S be a non empty set. Define an interval-valued fuzzy hyperoperation "$\circ$" on S by $\widetilde{x} \circ \widetilde{y} = \widetilde{\chi}_{\{x,y\}}$ for all $x, y \in S$, where $\widetilde{\chi}_{\{x,y\}}$ denotes the characteristic function of the set $\{x, y\}$. Then $(S, \circ)$ is an interval-valued fuzzy hypersemigroup.*

Proof: Let $a, x, y, z \in S$. If $a \in \{x, y, z\}$, then

$$\begin{aligned} ((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) &= \left(\widetilde{\chi}_{\{x,y\}} \circ \widetilde{z}\right)(a) \\ &= \bigvee_{s \in S} \left(\widetilde{\chi}_{\{x,y\}}(s) \wedge (\widetilde{s} \circ \widetilde{z})(a)\right) \\ &= \widetilde{\chi}_{\{x,y,z\}}(a) \\ &= \widetilde{1} \end{aligned}$$

and

$$\begin{aligned} (\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a) &= \left(\widetilde{x} \circ \widetilde{\chi}_{\{y,z\}}\right)(a) \\ &= \bigvee_{s \in S} \left((\widetilde{x} \circ \widetilde{s})(a) \wedge \widetilde{\chi}_{\{y,z\}}(s)\right) \end{aligned}$$

$$= \widetilde{\chi}_{\{x,y,z\}}(a)$$
$$= 1.$$

It follows that $(\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z} = \widetilde{x} \circ (\widetilde{y} \circ \widetilde{z})$. Assume that $a \notin \{x, y, z\}$. Similarly, we can show that

$$\begin{aligned} ((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) &= \widetilde{\chi}_{\{x,y,z\}}(a) \\ &= \widetilde{0} \\ &= \widetilde{\chi}_{\{x,y,z\}}(a) \\ &= (\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a). \end{aligned}$$

Hence, $(S, \circ)$ is an interval-valued fuzzy hypersemigroup. □

According to ([12], Example 2.6), if $(S, \cdot)$ is a semigroup, then $(S, \circ)$ is a fuzzy hypersemigroup. Now, we have

Theorem 2.2. *Let S be a semigroup. Define an interval-valued fuzzy hyperoperation "$\circ$" on S by $\widetilde{x} \circ \widetilde{y} = \widetilde{\chi}_{\{xy\}}$ for all $x, y \in S$, where $\widetilde{\chi}_{\{xy\}}$ denotes the characteristic function of the set $\{xy\}$. Then $(S, \circ)$ is an interval-valued fuzzy hypersemigroup.*

Proof: Let $a, x, y, z \in S$. Then

$$\begin{aligned} ((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) &= \left(\widetilde{\chi}_{\{xy\}} \circ \widetilde{z}\right)(a) \\ &= \bigvee_{s \in S} \left(\widetilde{\chi}_{\{xy\}}(s) \wedge (\widetilde{s} \circ \widetilde{z})(a)\right) \\ &= (\widetilde{xy} \circ \widetilde{z})(a) \\ &= \widetilde{\chi}_{\{(xy)z\}}(a) \\ &= \widetilde{\chi}_{\{x(yz)\}}(a) \\ &= (\widetilde{x} \circ \widetilde{yz})(a) \\ &= \bigvee_{s \in S} \left((\widetilde{x} \circ \widetilde{s})(a) \wedge \widetilde{\chi}_{\{yz\}}(s)\right) \\ &= \left(\widetilde{x} \circ \widetilde{\chi}_{\{yz\}}\right)(a) \\ &= (\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a). \end{aligned}$$

Hence, $(S, \circ)$ is an interval-valued fuzzy hypersemigroup. □

Note that if S is a semigroup, then S is an interval-valued fuzzy hypersemigroup. However, in general, the converse is not true as can be shown in the following example.

Example 2.1. *Let $(\{5, 6, 7, \ldots\}, \cdot)$ be a semigroup. By Theorem 2.2, $(\{5, 6, 7, \ldots\}, \circ)$ is an interval-valued fuzzy hypersemigroup which is not always a semigroup.*

In case of a semigroup with identity, we have the following result.

Corollary 2.1. *Let S be a semigroup with identity. Define an interval-valued fuzzy hyperoperation "$\circ$" on S by $\widetilde{x} \circ \widetilde{y} = \widetilde{\chi}_{\{xy\}}$ for all $x, y \in S$, where $\chi_{\{xy\}}$ denotes the characteristic function of the set $\{xy\}$. Then $(S, \circ)$ is an interval-valued fuzzy hypersemigroup with identity.*

Theorem 2.3. *Let S be a semigroup and $\widetilde{0} \neq \widetilde{\mu} \in \widetilde{\mathcal{F}(S)}$. Define an interval-valued fuzzy hyperoperation "$\circ$" on S by*

$$(\widetilde{x} \circ \widetilde{y})(a) = \begin{cases} \widetilde{\mu}(x) \wedge \widetilde{\mu}(y); & \text{if } a = xy \\ \widetilde{0}; & \text{otherwise} \end{cases}$$

for all $x, y \in S$. If $\widetilde{\mu}$ is an interval-valued fuzzy semigroup on S, then $(S, \circ)$ is an interval-valued fuzzy hypersemigroup.

Proof: Let $a, x, y, z \in S$. It is easy to see that,

$$\begin{aligned}((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) &= \bigvee_{s\in S} ((\widetilde{x} \circ \widetilde{y})(s) \wedge (\widetilde{s} \circ \widetilde{z})(a)) \\ &= (\widetilde{\mu}(x) \wedge \widetilde{\mu}(y)) \wedge (\widetilde{xy} \circ \widetilde{z})(a)\end{aligned}$$

and

$$\begin{aligned}(\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a) &= \bigvee_{s\in S} ((\widetilde{x} \circ \widetilde{s})(a) \wedge (\widetilde{y} \circ \widetilde{z})(s)) \\ &= (\widetilde{x} \circ \widetilde{yz})(a) \wedge (\widetilde{\mu}(y) \wedge \widetilde{\mu}(z))\end{aligned}$$

for all $a, x, y, z \in S$. If $a = (xy)z$, then $a = x(yz)$. By assumption,

$$\begin{aligned}((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) &= (\widetilde{\mu}(x) \wedge \widetilde{\mu}(y)) \wedge (\widetilde{xy} \circ \widetilde{z})(a) \\ &= \widetilde{\mu}(x) \wedge \widetilde{\mu}(y) \wedge \widetilde{\mu}(xy) \wedge \widetilde{\mu}(z) \\ &= \widetilde{\mu}(x) \wedge \widetilde{\mu}(y) \wedge \widetilde{\mu}(z) \\ &= \widetilde{\mu}(x) \wedge \widetilde{\mu}(yz) \wedge \widetilde{\mu}(y) \wedge \widetilde{\mu}(z) \\ &= (\widetilde{x} \circ \widetilde{yz})(a) \wedge (\widetilde{\mu}(y) \wedge \widetilde{\mu}(z)) \\ &= (\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a).\end{aligned}$$

Assume that $a \neq (xy)z$. Thus $a \neq x(yz)$. Clearly, $((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) = \widetilde{0} = (\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a)$. Hence, $(S, \circ)$ is an interval-valued fuzzy hypersemigroup. □

We continue this section with the following theorems.

Theorem 2.4. *Let $S = \boldsymbol{Z}^{-} \cup \{0, 1, \ldots, n\}$. Define an interval-valued fuzzy hyperoperation "$\circ$" on S by $\widetilde{x} \circ \widetilde{y} = \widetilde{\chi}_{\widetilde{x}\vee\widetilde{y}}$ for all $x, y \in S$. Then $(S, \circ)$ is an interval-valued fuzzy hypersemigroup.*

Proof: Let $a, x, y, z \in S$. Then

$$\begin{aligned}((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) &= \left(\widetilde{\chi}_{\widetilde{x}\vee\widetilde{y}} \circ \widetilde{z}\right)(a) \\ &= \bigvee_{s\in S} (\widetilde{\chi}_{\widetilde{x}\vee\widetilde{y}}(s) \wedge (\widetilde{s} \circ \widetilde{z})(a)) \\ &= ((\widetilde{x} \vee \widetilde{y}) \circ \widetilde{z})(a) \\ &= \widetilde{\chi}_{(\widetilde{x}\vee\widetilde{y})\vee\widetilde{z}}(a) \\ &= \widetilde{\chi}_{\widetilde{x}\vee(\widetilde{y}\vee\widetilde{z})}(a) \\ &= (\widetilde{x} \circ (\widetilde{y} \vee \widetilde{z}))(a) \\ &= \bigvee_{s\in S} ((\widetilde{x} \circ \widetilde{s})(a) \wedge \chi_{\widetilde{y}\vee\widetilde{z}}(s)) \\ &= \left(\widetilde{x} \circ \widetilde{\chi}_{\widetilde{y}\vee\widetilde{z}}\right)(a) \\ &= (\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a).\end{aligned}$$

Hence, $(S, \circ)$ is an interval-valued fuzzy hypersemigroup. □

Our main aim in the following is to study the notion of interval-valued fuzzy hypersemigroups.

Theorem 2.5. *Let $(S, \circ)$ be a fuzzy hypersemigroup. Define an interval-valued fuzzy hyperoperation "$\bullet$" on S by $\widetilde{x} \bullet \widetilde{y} = [0, x \circ y]$ for all $x, y \in S$. Then $(S, \bullet)$ is an interval-valued fuzzy hypersemigroup.*

Proof: Let $a, x, y, z \in S$. Then

$$\begin{aligned}((\widetilde{x} \bullet \widetilde{y}) \bullet \widetilde{z})(a) &= \bigvee_{s\in S} \left(\widetilde{x} \bullet \widetilde{y}\right)(s) \wedge \left(\widetilde{s} \bullet \widetilde{z}\right)(a) \\ &= \bigvee_{s\in S} [0, (x \circ y)(s)] \wedge [0, (s \circ z)(a)]\end{aligned}$$

$$\begin{aligned} &= [0, ((x \circ y) \circ z)(a)] \\ &= [0, (x \circ (y \circ z))(a)] \\ &= \bigvee_{s \in S} [0, (x \circ s)(a)] \wedge [0, (y \circ z)(s)] \\ &= \bigvee_{s \in S} (\widetilde{x} \bullet \widetilde{s})(a) \wedge (\widetilde{y} \bullet \widetilde{z})(s) \\ &= (\widetilde{x} \bullet (\widetilde{y} \bullet \widetilde{z}))(a). \end{aligned}$$

Hence, $(S, \bullet)$ is an interval-valued fuzzy hypersemigroup. □

Note that if S is a semigroup, then S is an interval-valued fuzzy hypersemigroup. However, in general, the converse is not true as can be shown in the following example.

Example 2.2. *Let $(\{5, 6, 7, \ldots\}, \cdot)$ be a semigroup. Define a fuzzy hyperoperation "$\circ$" on S by $x \star y = \chi_{\{xy\}}$ for all $x, y \in S$. It is easy to see that, $(S, \circ)$ is a fuzzy hypersemigroup (see [12]). By Theorem 2.5, $(\{5, 6, 7, \ldots\}, \bullet)$ is an interval-valued fuzzy hypersemigroup which is not always a semigroup.*

3. Product of Interval-Valued Fuzzy Hypergroupoids.

Now, we introduce the following useful concept.

Definition 3.1. *Let $(S, \circ)$ be an interval-valued fuzzy hypergroupoid and let $\widetilde{\mu}, \widetilde{\nu} \in \widetilde{\mathcal{F}(S)}$. The **product** $\widetilde{\mu} \circ \widetilde{\nu}$ is defined by $(\widetilde{\mu} \circ \widetilde{\nu})(a) = \bigvee_{x,y \in S} (\widetilde{\mu}(x) \wedge (\widetilde{x} \circ \widetilde{y})(a) \wedge \widetilde{\nu}(y))$ for all $a \in S$.*

The following result holds.

Theorem 3.1. *Let $(S, \circ)$ be an interval-valued fuzzy hypersemigroup. Then the following statements hold:*

1) $\widetilde{\chi}_x \circ \widetilde{\chi}_y = \widetilde{x} \circ \widetilde{y}$ for all $x, y \in S$.

2) For every $x \in S$, $\widetilde{\chi}_S \circ \widetilde{x} = \widetilde{S} \circ \widetilde{x}$ and $\widetilde{x} \circ \widetilde{\chi}_S = \widetilde{x} \circ \widetilde{S}$.

3) For every $\widetilde{\mu} \in \widetilde{\mathcal{F}(S)}$, $\widetilde{\chi}_S \circ \widetilde{\mu} = \widetilde{S} \circ \widetilde{\mu}$ and $\widetilde{\mu} \circ \widetilde{\chi}_S = \widetilde{\mu} \circ \widetilde{S}$.

Proof: 1) Let $a, x, y \in S$. Then

$$\begin{aligned} \widetilde{\chi}_x \circ \widetilde{\chi}_y(a) &= \bigvee_{r,s \in S} (\widetilde{\chi}_x(r) \wedge (\widetilde{r} \circ \widetilde{s})(a) \wedge \widetilde{\chi}_y(s)) \\ &= \widetilde{1} \wedge (\widetilde{x} \circ \widetilde{y})(a) \wedge \widetilde{1} \\ &= (\widetilde{x} \circ \widetilde{y})(a). \end{aligned}$$

Hence, $\widetilde{\chi}_x \circ \widetilde{\chi}_y = \widetilde{x} \circ \widetilde{y}$.

2) Let $a, x \in S$. Then

$$\begin{aligned} (\widetilde{\chi}_S \circ \widetilde{x})(a) &= \bigvee_{s \in S} (\widetilde{\chi}_S(s) \wedge (\widetilde{s} \circ \widetilde{x})(a)) \\ &= \bigvee_{s \in S} \left(\widetilde{1} \wedge (\widetilde{s} \circ \widetilde{x})(a)\right) \\ &= \bigvee_{s \in S} (\widetilde{s} \circ \widetilde{x})(a) \\ &= \left(\widetilde{S} \circ \widetilde{x}\right)(a). \end{aligned}$$

Hence, $\widetilde{\chi}_S \circ \widetilde{x} = \widetilde{S} \circ \widetilde{x}$. Similarly we can show that $\widetilde{x} \circ \widetilde{\chi}_S = \widetilde{x} \circ \widetilde{S}$.

3) Let $\widetilde{\mu} \in \mathcal{F}(S)$ and $a \in S$. Then

$$\begin{aligned}(\widetilde{\chi}_S \circ \widetilde{\mu})(a) &= \bigvee_{r,s\in S} (\widetilde{\chi}_S(r) \wedge (\widetilde{r} \circ \widetilde{s})(a) \wedge \widetilde{\mu}(s)) \\ &= \bigvee_{r,s\in S} \left(\widetilde{1} \wedge (\widetilde{r} \circ \widetilde{s})(a) \wedge \widetilde{\mu}(s)\right) \\ &= \bigvee_{r,s\in S} ((\widetilde{r} \circ \widetilde{s})(a) \wedge \widetilde{\mu}(s)) \\ &= \bigvee_{r\in S} ((\widetilde{r} \circ \widetilde{\mu})(a)) \\ &= \left(\widetilde{S} \circ \widetilde{\mu}\right)(a).\end{aligned}$$

Hence, $\widetilde{\chi}_S \circ \widetilde{\mu} = \widetilde{S} \circ \widetilde{\mu}$. Similarly we can show that $\widetilde{\mu} \circ \widetilde{\chi}_S = \widetilde{\mu} \circ \widetilde{S}$. □

Furthermore, we have the following theorem.

Theorem 3.2. *Let S be an interval-valued fuzzy hypersemigroup and $\widetilde{\mu}, \widetilde{\nu}, \widetilde{\lambda} \in \widetilde{\mathcal{F}(S)}$. Then the following statements hold:*

1) For every $x, y \in S$, $(\widetilde{x} \circ \widetilde{y}) \circ \widetilde{\mu} = \widetilde{x} \circ (\widetilde{y} \circ \widetilde{\mu})$.
2) For every $x, y \in S$, $(\widetilde{x} \circ \widetilde{\mu}) \circ \widetilde{y} = \widetilde{x} \circ (\widetilde{\mu} \circ \widetilde{y})$.
3) For every $x, y \in S$, $(\widetilde{\mu} \circ \widetilde{x}) \circ \widetilde{y} = \widetilde{\mu} \circ (\widetilde{x} \circ \widetilde{y})$.
4) For every $x \in S$, $(\widetilde{\mu} \circ \widetilde{\nu}) \circ \widetilde{x} = \widetilde{\mu} \circ (\widetilde{\nu} \circ \widetilde{x})$.
5) For every $x \in S$, $(\widetilde{\mu} \circ \widetilde{x}) \circ \widetilde{\nu} = \widetilde{\mu} \circ (\widetilde{x} \circ \widetilde{\nu})$.
6) For every $x \in S$, $(\widetilde{x} \circ \widetilde{\mu}) \circ \widetilde{\nu} = \widetilde{x} \circ (\widetilde{\mu} \circ \widetilde{\nu})$.
7) $(\widetilde{\mu} \circ \widetilde{\nu}) \circ \widetilde{\lambda} = \widetilde{\mu} \circ \left(\widetilde{\nu} \circ \widetilde{\lambda}\right)$.

Proof: 1) Let $a, x, y \in S$ and $\widetilde{\mu} \in \widetilde{\mathcal{F}(S)}$. Then

$$\begin{aligned}((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{\mu})(a) &= \bigvee_{r\in S} (((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{r})(a) \wedge \widetilde{\mu}(r)) \\ &= \bigvee_{r\in S} ((\widetilde{x} \circ (\widetilde{y} \circ \widetilde{r}))(a) \wedge \widetilde{\mu}(r)) \\ &= \bigvee_{r,s\in S} ((\widetilde{x} \circ \widetilde{s})(a) \wedge (\widetilde{y} \circ \widetilde{r})(s) \wedge \widetilde{\mu}(r)) \\ &= \bigvee_{r,s\in S} ((\widetilde{x} \circ \widetilde{s})(a) \wedge (\widetilde{y} \circ \widetilde{\mu})(s)) \\ &= (\widetilde{x} \circ (\widetilde{y} \circ \widetilde{\mu}))(a).\end{aligned}$$

Hence, $(\widetilde{x} \circ \widetilde{y}) \circ \widetilde{\mu} = \widetilde{x} \circ (\widetilde{y} \circ \widetilde{\mu})$.

2)-7) The proof is similar to 1). □

Remark 3.1. *Let S be an interval-valued fuzzy hypersemigroup. Then by Theorem 3.2, $\left(\widetilde{\mathcal{F}(S)}, \circ\right)$ is a semigroup.*

In case of an interval-valued fuzzy hypersemigroup with identity, we have the following result.

Corollary 3.1. *Let S be an interval-valued fuzzy hypersemigroup with identity. Then the following statements hold:*

1) $\widetilde{e} \circ \widetilde{\mu} = \widetilde{\mu}$ and $\widetilde{\mu} \circ \widetilde{e} = \widetilde{\mu}$ for all $\widetilde{\mu} \in \widetilde{\mathcal{F}(S)}$.
2) $\widetilde{e} \circ (\widetilde{x} \circ \widetilde{y}) = \widetilde{x} \circ \widetilde{y}$ and $(\widetilde{x} \circ \widetilde{y}) \circ \widetilde{e} = \widetilde{x} \circ \widetilde{y}$ for all $x, y \in S$.
3) $\widetilde{x} \circ (\widetilde{e} \circ \widetilde{y}) = \widetilde{x} \circ \widetilde{y}$ and $\widetilde{x} \circ (\widetilde{y} \circ \widetilde{e}) = \widetilde{x} \circ \widetilde{y}$ for all $x, y \in S$.

Proof: 1) Let $a \in S$ and $\widetilde{\mu} \in \widetilde{\mathcal{F}(S)}$. Then

$$\begin{aligned}(\widetilde{e} \circ \widetilde{\mu})(a) &= \bigvee_{r \in S} ((\widetilde{e} \circ \widetilde{r})(a) \wedge \widetilde{\mu}(r)) \\ &= \widetilde{1} \wedge \widetilde{\mu}(a) \\ &= \widetilde{\mu}(a).\end{aligned}$$

Hence, $\widetilde{e} \circ \widetilde{\mu} = \widetilde{\mu}$. Similarly we can show that $\widetilde{\mu} \circ \widetilde{e} = \widetilde{\mu}$.

2) The proof is similar to part 1).

3) Let $a, x, y \in S$. Then

$$\begin{aligned}(\widetilde{x} \circ (\widetilde{e} \circ \widetilde{y}))(a) &= \bigvee_{s \in S} ((\widetilde{x} \circ \widetilde{s})(a) \wedge (\widetilde{e} \circ \widetilde{y})(s)) \\ &= (\widetilde{x} \circ \widetilde{y})(a) \wedge \widetilde{1} \\ &= (\widetilde{x} \circ \widetilde{y})(a).\end{aligned}$$

Hence, $\widetilde{x} \circ (\widetilde{e} \circ \widetilde{y}) = \widetilde{x} \circ \widetilde{y}$. Similarly we can show that $\widetilde{x} \circ (\widetilde{y} \circ \widetilde{e}) = \widetilde{x} \circ \widetilde{y}$. □

Let S be a semihypergroup, endowed with an interval-valued fuzzy hypersemigroups "$\circ$" and for all $x, y \in S$, consider the $\widetilde{\alpha}$-cuts

$$(x \circ y)_{\widetilde{\alpha}} := \{s \in S : (\widetilde{x} \circ \widetilde{y})(s) \geq \widetilde{\alpha}\}$$

of $\widetilde{x} \circ \widetilde{y}$, where $\widetilde{\alpha} \in D[0,1]$. For all $\widetilde{\alpha} \in D[0,1]$, we define the following crisp hyperoperation on S: $x \circ_{\widetilde{\alpha}} y := (x \circ y)_{\widetilde{\alpha}}$.

Theorem 3.3. *Let S be a semihypergroup and $x \in S$. Then $\widetilde{\chi}_S = \widetilde{x} \circ \widetilde{S}$ if and only if $S = x \circ_{\widetilde{\alpha}} S$ for all $\widetilde{\alpha} \in D[0,1]$.*

Proof: Suppose that $\widetilde{\chi}_S = \widetilde{x} \circ \widetilde{S}$ for all $x \in S$. Let $a \in S$ and $\widetilde{\alpha} \in D[0,1]$. By assumption,

$$\begin{aligned}\bigvee_{s \in S} (\widetilde{x} \circ \widetilde{s})(a) &= \widetilde{\chi}_S(a) \\ &= \widetilde{1} \\ &\geq \widetilde{\alpha}.\end{aligned}$$

Then there exists $r \in S$ such that $(\widetilde{x} \circ \widetilde{r})(a) \geq \widetilde{\alpha}$, which means that $a \in x \circ_{\widetilde{\alpha}} r$. Hence, $S = x \circ_{\widetilde{\alpha}} S$.

Conversely assume that, $S = x \circ_{\widetilde{\alpha}} S$ for all $\widetilde{\alpha} \in D[0,1]$. By assumption, $S = x \circ_{\widetilde{1}} S$. Then there exists $s \in S$ such that $a \in x \circ_{\widetilde{1}} s$ for all $a \in S$, which means that $(\widetilde{x} \circ \widetilde{s})(a) = \widetilde{1}$. Consequently, $\widetilde{\chi}_S = \widetilde{x} \circ \widetilde{S}$. □

Now the following theorem is one of the prominent characterization of the interval-valued fuzzy semihypergroup.

Theorem 3.4. *Let S be a non empty set and $x, y, z \in S$, $\widetilde{\alpha} \in D[0,1]$. Then the following statements hold:*

1) $((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) \geq \widetilde{\alpha}$ if and only if $a \in (\widetilde{x} \circ_{\widetilde{\alpha}} \widetilde{y}) \circ_{\widetilde{\alpha}} \widetilde{z}$.

2) $(\widetilde{x} \circ (\widetilde{y} \circ \widetilde{z}))(a) \geq \widetilde{\alpha}$ if and only if $a \in \widetilde{x} \circ_{\widetilde{\alpha}} (\widetilde{y} \circ_{\widetilde{\alpha}} \widetilde{z})$.

Proof: 1) Let $a, x, y, z \in S$ and $\widetilde{\alpha} \in D[0,1]$. It is easy to see that

$$\begin{aligned}((\widetilde{x} \circ \widetilde{y}) \circ \widetilde{z})(a) &= \bigvee_{s \in S} ((\widetilde{x} \circ \widetilde{y})(s) \wedge (\widetilde{s} \circ \widetilde{z})(a)) \\ &\geq \widetilde{\alpha}\end{aligned}$$

if and only if there exists $r \in S$ such that $r \in x \circ_{\alpha} y$ and $a \in r \circ_{\widetilde{\alpha}} z$, which means that $(\widetilde{x} \circ \widetilde{y})(r) \geq \widetilde{\alpha}$ and $(\widetilde{r} \circ \widetilde{z})(a) \geq \widetilde{\alpha}$. This completes the proof.

2) The proof is similar to part 1). □

Now by using the above Theorem 3.4, we can easily prove the following results.

Theorem 3.5. *$(S, \circ)$ is an interval-valued fuzzy semihypergroup if and only if $(S, \circ_{\widetilde{\alpha}})$ is a semihypergroup for all $\widetilde{\alpha} \in D[0,1]$.*

4. **Conclusion.** In this study the structure of interval-valued fuzzy hypersemigroups with special properties always plays an important role. Moreover, we define interval-valued fuzzy subhypersemigroups, interval-valued fuzzy hyperideal (right interval-valued fuzzy hyperideal, left fuzzy hyperideal), interval-valued fuzzy hyper bi-ideals and interval-valued fuzzy hyper interior ideals in interval-valued fuzzy hypersemigroups and some properties of them are obtained.

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