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TWO-STAGE BAYESIAN FINITE MEMORY STRUCTURE SMOOTHER FOR DISCRETE-TIME SYSTEMS

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Abstract. *This paper addresses a two-stage finite memory structure (FMS) smoother using the chi-square test statistic for discrete-time systems with a control input. An alternative FMS smoother is firstly developed from the conditional density of the current state given finite past measurements, which is called the Bayesian FMS smoother. According to the presence or absence of uncertainty, one of the primary Bayesian FMS smoother or the secondary Bayesian FMS smoother works selectively to obtain a valid estimate. In order to indicate presence or absence of uncertainty, operate the suitable one from two smoothers, and then obtain the valid smoothing estimate, declaration rule and test variables are defined. Simulation results validate the effectiveness of the proposed two-stage Bayesian FMS smoother.*
Keywords: Bayesian estimation, Finite memory structure, Smoother, Temporary uncertainty, Test variable

1. **Introduction.** Finite memory structure (FMS) filter [1,2] and smoother [3-5] have been applied successfully for various engineering problems. Even if FMS filter and smoother can show greater noise suppression as the window length increases, the tracking speed of the state estimate for the actual state variable worsens in proportion to the window length, which can degrade the estimation performance of FMS filter and smoother. This implies FMS filter and smoother require a compromise between the noise suppression and the tracking speed of the state estimate. According to this observation, the estimation error of FMS filter and smoother with a short measurement window length is smaller than that of the FMS filter with a long measurement window length, while temporary uncertainty exists. In addition, the convergence of the estimation error for FMS filter and smoother with a short window length is much faster than that of FMS filter and smoother with a long window length when temporary uncertainty is disappearing. This means that FMS filter and smoother with a short window length are superior in terms of the tracking ability. Thus, if FMS filter and smoother with a short window length are applied to temporarily uncertain systems, they can outperform FMS filter and smoother with a long window length, although the robustness is not considered in the designed process.

To verify the above observation, the two-stage estimation using two FMS filters with different measurement windows was developed [6]. However, the two-stage estimation approach using FMS smoothers has not been addressed so far. The FMS smoother in [3-5] is known to have the following common advantages. The smoother generally utilizes more measurement information than the filter to provide state estimates, which can give more accurate estimation performance than the filter. In addition, since the smoother

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provides state estimates at the delayed time using measurement information up to the current time, measurement information can be reflected in advance in the presence of the state change, which can give more fast convergence than the filter. If these superiorities of the FMS smoother can be verified in practical applications, this might be very informative for engineers and researchers in control and estimation areas, which is a main motivation of this paper. In addition, the control input term has not been considered in the existing two-stage estimation [6]. Actually, the state-space model with control input can be often used for various control engineering problems such as electric motor system, and automotive suspension system [5,7]. Moreover, it would be very meaningful if the Bayesian estimation strategy was considered to the FMS smoother design just like the FMS filter design of [6,8].

Therefore, in this paper, a two-stage FMS smoother with two kinds of measurement windows is proposed using the chi-square test statistic for discrete-time systems with a control input. Firstly, an alternative FMS smoother, called the Bayesian FMS smoother, is developed from the conditional density of the current state given finite past measurements. Then, one of the two smoothers, the primary Bayesian FMS smoother and the secondary Bayesian FMS smoother, with different measurement windows is operated selectively to obtain the valid estimate according to presence or absence of uncertainty. The primary Bayesian FMS smoother is selected for the nominal system and the secondary Bayesian FMS smoother is selected for the temporarily uncertain system, respectively. A declaration rule is defined to indicate the presence or absence of uncertainty, operate the suitable one from two smoothers, and then obtain the valid smoothing estimate. A test variable for the declaration rule is developed using a chi-square test statistic from the estimation error and compared with a precomputed threshold. Finally, computer simulations are performed for an electric motor system to verify the proposed two-stage Bayesian FMS smoother and compare with the standard FMS smoother as well as the existing two-stage FMS filter. Through computer simulation works, it is shown that the proposed smoother works well for the nominal system as well as the temporarily uncertain system. It is also shown that the proposed smoother can be remarkably better than the existing two-stage FMS filter for the temporarily uncertain system.

This paper is organized as follows. In Section 2, the Bayesian FMS smoother from conditional density of lagged state is developed. In Section 3, the two-stage Bayesian FMS smoother is proposed. In Section 4, computer simulations are performed. Finally, conclusions are presented in Section 5.

2. Bayesian Finite Memory Structure Smoother from Conditional Density of Lagged State.

A general discrete-time state-space model with a control input is considered as follows:

$$x_{i+1} = Ax_i + Bu_i + Gw_i, \tag{1}$$

$$z_i = Cx_i + v_i, \tag{2}$$

where $x_i \in \Re^n$ is the unknown state vector, $u_i \in \Re^n$ is the control input vector, and $z_i \in \Re^q$ is the sensor measurement vector. The state vector x_{i_0} at the initial time i_0 of system is a random variable with a mean $\bar{x}_{i_0}$ and a covariance P_{i_0}. A dynamic system can often contain noises such as the system noise $w_i \in \Re^p$ and the measurement noise $v_i \in \Re^q$. These noises are random variables with zero-mean white Gaussian and are mutually uncorrelated. In addition, these noises are also uncorrelated with the initial state vector x_{i_0}. The covariances of noises w_i and v_i are denoted by Q and R, respectively and they are assumed to be positive definite matrices.

In this section, an alternative FMS smoother to estimate the state x_{i-d} at the lagged time $i-d$ is developed using only finite measurements as well as inputs on the most recent window $[i-M, i]$. The lagged time $i-d$ means there is a fixed delay between the measurement and the availability of its estimate. The positive integer d is the delay length satisfying $0 \leq d < M$ and equal to the number of discrete time steps between the lagged time $i-d$ at which the state is to be estimated and the current time i of the last measurement used in estimating it. Finite measurements and inputs on the most recent window $[i-M, i]$ are denoted by Z_i and U_i, respectively, and represented by

$$Z_M \triangleq \left[z_{i-M}^T \; z_{i-M+1}^T \ldots z_{i-1}^T\right]^T, \quad U_M \triangleq \left[u_{i-M}^T \; u_{i-M+1}^T \ldots u_{i-1}^T\right]^T. \tag{3}$$

Using Z_M and U_M, the discrete-time state-space model (1) and (2) can be represented in the following regression form

$$Z_M - \bar{\Xi}_M U_M = \bar{\Gamma}_M x_{i-M} + \bar{\Lambda}_M W_M + V_M, \tag{4}$$

where W_M and V_M have the same form as (3) for w_i, v_i, respectively, and matrices $\bar{\Gamma}_M$, $\bar{\Xi}_M$, $\bar{\Lambda}_M$ are as follows:

$$\bar{\Gamma}_M \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{M-2} \\ CA^{M-1} \end{bmatrix}, \quad \bar{\Xi}_M \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CB & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{M-3}B & CA^{M-4}B & \cdots & 0 & 0 \\ CA^{M-2}B & CA^{M-3}B & \cdots & CB & 0 \end{bmatrix},$$

$$\bar{\Lambda}_M \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CG & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{M-3}G & CA^{M-4}G & \cdots & 0 & 0 \\ CA^{M-2}G & CA^{M-3}G & \cdots & CG & 0 \end{bmatrix}.$$

From the discrete-time state-space model (1) and (2), the state x_{i-d} at the lagged time $i-d$ is represented by

$$x_{i-d} = A^{M-d} x_{i_M} + \tilde{\Xi}_M U_M + \tilde{\Lambda}_M W_M, \tag{5}$$

where

$$\tilde{\Xi}_M \triangleq \left[A^{M-d-1}B \; \cdots \; AB \; B \; \overbrace{0 \; 0 \; \cdots \; 0}^{d} \right], \; \tilde{\Lambda}_M \triangleq \left[A^{M-d-1}G \; \cdots \; AG \; G \; \overbrace{0 \; 0 \; \cdots \; 0}^{d} \right]. \tag{6}$$

Therefore, using (5), the regression form (4) can be expressed in terms with x_{i-d} at the lagged time $i-d$ as follows:

$$Z_M - \Xi_M U_M = \Gamma_M x_{i-d} + \Lambda_M W_M + V_M, \tag{7}$$

where

$$\begin{aligned} &\Gamma_M \triangleq \bar{\Gamma}_M A^{-(M-d)}, \quad \Lambda_M \triangleq \bar{\Lambda}_M - \bar{\Gamma}_M A^{-(M-d)} \tilde{\Lambda}_M, \\ &\Xi_M \triangleq \bar{\Xi}_M - \bar{\Gamma}_M A^{-(M-d)} \tilde{\Xi}_M. \end{aligned} \tag{8}$$

The noise term $\Lambda_M W_M + V_M$ in (7) is zero-mean white Gaussian as follows:

$$\Lambda_M W_M + V_M \sim \mathcal{N}\big(Z_M - \Xi_M U_M; 0, \Pi_M\big), \tag{9}$$

where $\mathcal{N}\big(Z_M - \Xi_M U_M; 0, \Pi_M\big)$ denotes the Gaussian probability density function (pdf) evaluated at Z_M with zero-mean and covariance matrix

$$\Pi_M \triangleq \Lambda_M \left[\mathrm{diag}(\overbrace{Q \; Q \; Q \; \cdots \; Q}^{M}) \right] \Lambda_M{}^T + \left[\mathrm{diag}(\overbrace{R \; R \; R \; \cdots \; R}^{M}) \right], \tag{10}$$

where $\mathrm{diag}(Q \; Q \; Q \; \cdots \; Q)$ and $\mathrm{diag}(R \; R \; R \; \cdots \; R)$ denote block-diagonal matrices with M elements of Q and R, respectively.

As shown in Bayesian estimation strategy [2,6], the FMS smoother can be interested in the pdf that is conditional on a finite past measurements and inputs $Z_M - \Xi_M U_M$ on the most recent window $[i-M, i]$. The most recent window $[i-M, i]$ becomes the averaging window of M points. To develop an alternative FMS smoother, called the Bayesian FMS smoother, the conditional density of the state x_{i-d} at the lagged time $i-d$ given finite measurements and inputs $Z_M - \Xi_M U_M$ is derived.

Theorem 2.1. *From the linearity described in (7), the conditional density of state x_{i-d} at the lagged time $i-d$ given finite measurements and inputs $Z_M - \Xi_M U_M$ has the following expression:*

$$p\left(x_{i-d}|Z_M - \Xi_M U_M\right) = \mathcal{N}\left(x_{i-d}; \hat{x}_{i-d}, \Sigma_M\right), \tag{11}$$

where $\mathcal{N}\left(x_{i-d}; \hat{x}_{i-d}, \Sigma_M\right)$ denotes the Gaussian pdf evaluated at x_{i-d} with mean $\hat{x}_{i-d}$ and covariance matrix Σ_M as follows:

$$\hat{x}_{i-d} = \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} \Gamma_M^T \Pi_M^{-1} (Z_M - \Xi_M U_M), \quad \Sigma_M = \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1}. \tag{12}$$

Proof: Equation (7) can be represented by

$$\Gamma_M x_{i-d} = (Z_M - \Xi_M U_M) - (\Lambda_M W_M + V_M). \tag{13}$$

Then, multiplying both sides of (13) by

$$\left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} \Gamma_M^T \Pi_M^{-1}$$

leads to

$$x_{i-d} = \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} \Gamma_M^T \Pi_M^{-1} [(Z_M - \Xi_M U_M) - (\Lambda_M W_M + V_M)]. \tag{14}$$

Hence, for given finite measurements and inputs $Z_M - \Xi_M U_M$, Equation (14) clearly means that the state x_{i-d} at the lagged time $i-d$ is a multi-variate Gaussian with its mean

$$\hat{x}_{i-d} = \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} \Gamma_M^T \Pi_M^{-1} (Z_M - \Xi_M U_M),$$

and covariance

$$\begin{aligned}\Sigma_M &= \left[\left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} \Gamma_M^T \Pi_M^{-1}\right] \Pi_M \left[\Pi_M^{-1} \Gamma_M \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1}\right] \\ &= \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right) \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} = \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1}.\end{aligned}$$

Therefore, the conditional density of state x_{i-d} at the lagged time $i-d$ given finite measurements and inputs $Z_M - \Xi_M U_M$ has the following expression:

$$p(x_{i-d}|Z_M - \Xi_M U_M) = \mathcal{N}(x_{i-d}; \hat{x}_{i-d}, \Sigma_M).$$

This completes the proof. □

Therefore, from the conditional density (11) of state x_{i-d} at the lagged time $i-d$, the Bayesian FMS smoother with the following simple matrix form

$$\hat{x}_{i-d} = \left(\Gamma_M^T \Pi_M^{-1} \Gamma_M\right)^{-1} \Gamma_M^T \Pi_M^{-1} (Z_M - \Xi_M U_M) \tag{15}$$

provides the state estimate $\hat{x}_{i-d}$ conditional on finite measurements and inputs $Z_M - \Xi_M U_M$.

3. **Two-Stage Bayesian FMS Smoother.** To deal with temporary uncertainties, how to get a proper measurement window length M for the Bayesian FMS smoother might be an important issue. The window length affects differently the performance of the FMS smoother according to presence or absence of temporary uncertainties. FMS filter and smoother are well known to have better noise suppression as the window length grows. Hence, the noise suppression of the Bayesian FMS smoother can be closely related to the window length. However, even if the Bayesian FMS smoother can show greater noise suppression as the window length increases, the tracking speed of state estimate for actual state variable worsens in proportion to the window length, which can degrade the

estimation performance of the Bayesian FMS smoother. This means that the Bayesian FMS smoother requires a compromise between the noise suppression and the tracking speed of the state estimate.

According to the above observation, the estimation error of the Bayesian FMS smoother with a short window length is smaller than that of the Bayesian FMS smoother with a long window length while uncertainty exists. In addition, the convergence of the estimation error for the Bayesian FMS smoother with a short window length is much faster than that of the Bayesian FMS smoother with a long window length when temporary uncertainty is disappearing. This means that the Bayesian FMS smoother with a short window length is superior in terms of the tracking ability. Thus, although the Bayesian FMS smoother with a short window length is designed without considering the robustness, the Bayesian FMS smoother with a short window length can outperform the Bayesian FMS smoother with a long window length when applied to temporarily uncertain systems. Meanwhile, the Bayesian FMS smoother with a long window length can be better than the Bayesian FMS smoother with a short window length for the nominal system where temporary uncertainty completely disappears or there is no temporary uncertainty.

In this section, the two-stage Bayesian FMS smoother is proposed to cover the nominal system as well as the temporarily uncertain system by applying two kinds of Bayesian FMS smoothers selectively. Two kinds of Bayesian FMS smoothers are defined by the primary Bayesian FMS smoother with long window length M_p and the secondary Bayesian FMS smoother with short window length M_s. That is, the window length M_p is larger than the window length M_s.

Using the simple matrix form (15) for the Bayesian FMS smoother, the primary Bayesian FMS smoother is denoted by $\hat{x}^p_{i-d}$ and has the window length M_p as follows:

$$\hat{x}^p_{i-d} = \left(\Gamma^T_{M_p}\Pi^{-1}_{M_p}\Gamma_{M_p}\right)^{-1}\Gamma^T_{M_p}\Pi^{-1}_{M_p}\left(Z_{M_p} - \Xi_{M_p}U_{M_p}\right), \tag{16}$$

and the secondary Bayesian FMS smoother is denoted by $\hat{x}^s_{i-d}$ and has the window length M_s as follows:

$$\hat{x}^s_{i-d} = \left(\Gamma^T_{M_s}\Pi^{-1}_{M_s}\Gamma_{M_s}\right)^{-1}\Gamma^T_{M_s}\Pi^{-1}_{M_s}\left(Z_{M_s} - \Xi_{M_s}U_{M_s}\right), \tag{17}$$

where Γ_{M_p}, Π_{M_p}, $Z_{M_p} - \Xi_{M_p}U_{M_p}$, Γ_{M_s}, Π_{M_s} and $Z_{M_s} - \Xi_{M_s}U_{M_s}$ can be obtained from (8) and (10). Matrices $\left(\Gamma^T_{M_p}\Pi^{-1}_{M_p}\Gamma_{M_p}\right)^{-1}\Gamma^T_{M_p}\Pi^{-1}_{M_p}$ in (16) and $\left(\Gamma^T_{M_s}\Pi^{-1}_{M_s}\Gamma_{M_s}\right)^{-1}\Gamma^T_{M_s}\Pi^{-1}_{M_s}$ in (17) need only one computation on the interval $[0, M_p]$ and $[0, M_s]$, respectively, once. And then, they are time-invariant for all moving windows. Thus, two FMS Bayesian smoothers $\hat{x}^p_{i-d}$ (16) and $\hat{x}^s_{i-d}$ (17) are time-invariant.

One of the two Bayesian FMS smoothing estimates is selected as the valid estimate according to presence or absence of uncertainty. The primary Bayesian FMS smoother $\hat{x}^p_{i-d}$ is selected as the valid estimate $\hat{x}_{i-d}$ for the nominal system and the secondary Bayesian FMS smoother $\hat{x}^s_{i-d}$ is selected as the valid estimate $\hat{x}_{i-d}$ for the temporarily uncertain system as follows:

$$\hat{x}_{i-d} = \begin{cases} \hat{x}^p_{i-d} & \text{in case of nominal system,} \\ \hat{x}^s_{i-d} & \text{in case of temporarily uncertain system.} \end{cases}$$

In order to indicate presence or absence of uncertainty, operate the suitable one from two smoothers, and then obtain the valid smoothing estimate, a declaration rule is defined. The declaration rule determines two declaration cases of uncertainty presence and absence. The uncertainty presence indicates that the uncertainty occurs from the nominal system. On the other hand, the uncertainty absence indicates that the uncertainty is gone. A test variable t_i required for the uncertainty presence and absence declaration is formulated by the estimation error of the primary Bayesian FMS smoother $\hat{x}^p_{i-d}$ as follows:

$$t_{i-d} = \left(x_{i-d} - \hat{x}^p_{i-d}\right)^T \Sigma^{-1}_{M_p} \left(x_{i-d} - \hat{x}^p_{i-d}\right). \quad (18)$$

The matrix $\Sigma^{-1}_{M_p}$ is the covariance of $x_{i-d} - \hat{x}^p_{i-d}$ and obtained from (12). Since the estimation error $x_{i-d} - \hat{x}^p_{i-d}$ is in Gaussian distribution, the test variable (18) is in the chi-squared distribution with one degree of freedom. The chi-square, also written as χ^2, test statistic was used for abnormal signal detection [9,10]. As shown in (18), a chi-square test statistic is developed from the difference between a state and its smoothing estimate and then compared with a precomputed threshold for uncertainty presence and absence declaration.

The test variable t_{i-d} increases from the chi-squared distribution in proportion to the power of the uncertainty if an uncertainty appears. On the other hand, the test variable t_{i-d} decreases from the chi-squared distribution in proportion to the power of the uncertainty if an uncertainty disappears. Hence, comparing the test variable t_{i-d} to a threshold value γ can declare the presence or absence of uncertainty.

A threshold value is precomputed to compare with the test variable. The threshold value is set relatively to the sensitivity of the estimation error $x_{i-d} - \hat{x}^p_{i-d}$. That is, a too low threshold value causes an excessive false alarm rate, on the other hand, a too high one brings about insensitive uncertainty presence declaration. Hence, a threshold value can be precomputed from the chi-squared distribution function with the consideration of rational probability false alarm (PFA) because the test variable (18) forms a chi-squared distribution. The relationship between the threshold value and the PFA is represented by the following one degree of freedom chi-squared distribution function:

$$PFA = 1 - P_{\chi^2}(\gamma_*) = 1 - \frac{1}{2.5066} \int_0^{\gamma_*} \varepsilon^{-1/2} e^{-\varepsilon/2} d\varepsilon,$$

where γ_* stands for the threshold value.

Thus, when $t_{i-d} > \gamma_*$, the secondary Bayesian FMS smoother $\hat{x}^s_{i-d}$ is selected as the valid estimate $\hat{x}_{i-d}$, which indicates that the uncertainty occurs. And then, when $t_{i-d} < \gamma_*$, the primary Bayesian FMS smoother $\hat{x}^p_{i-d}$ is selected as the valid estimate $\hat{x}_{i-d}$, which indicates that the uncertainty disappears as follows:

$$\hat{x}_{i-d} = \begin{cases} \hat{x}^p_{i-d} & \text{if } t_{i-d} \le \gamma_* \text{ (uncertainty absence)}, \\ \hat{x}^s_{i-d} & \text{if } t_{i-d} > \gamma_* \text{ (uncertainty presence)}. \end{cases} \quad (19)$$

4. **Computer Simulations.** To verify the applicability of the proposed two-stage Bayesian FMS smoother with two kinds of measurement windows and to compare it with the standard FMS smoother with one measurement window as well as the existing two-stage FMS filter, extensive computer simulations using the well-known commercial software Matlab are performed for an electric motor system.

The discrete-time nominal direct current electric motor model without model uncertainty is given as follows [7]:

$$A = \begin{bmatrix} 0.8178 & -0.0011 \\ 0.0563 & 0.3678 \end{bmatrix}, \; G = \begin{bmatrix} 0.0006 & 0 \\ 0 & 0.0057 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad (20)$$

where the electric motor is assumed to be operated without any payload. The electric motor encounters the input voltage to drive the motor as an external source which is treated as a control input and emulated by the unit step for simulations. Covariances for system and measurement noises are taken as $Q = 0.01^2 I_{2\times 2}$ and $R = 0.01^2$, respectively. Two kinds of measurement window lengths are taken as $M_p = 20$ and $M_s = 10$, respectively. The lagged length is taken as $d = 3$. Simulations of 20 runs are performed using different system and measurement noises to make the comparison clearer. Each single simulation run lasts 600 samples.

A model uncertainty is considered as a temporary uncertainty. Thus, the actual state-space model for the electric motor system becomes

$$\bar{A} = A + \Delta A, \quad \bar{C} = C + \Delta C, \tag{21}$$

where ΔA and ΔC for the electric motor system are emulated by

$$\Delta A = \begin{bmatrix} \delta_i & 0 \\ 0 & \delta_i \end{bmatrix}, \quad \Delta C = \begin{bmatrix} 0.1\delta_i & 0 \end{bmatrix}, \quad \delta_i = \begin{cases} 0.05 & \text{if } 200 \leq i \leq 350, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Hence, although two FMS smoothers $\hat{x}^p_{i-d}$ (16) and $\hat{x}^s_{i-d}$ (17) are designed for the nominal state-space model (20) with A and C, they are applied actually for the temporarily uncertain system (21) with model uncertainties (22).

The threshold value is set to $\gamma = 7.88$ corresponding to PFA $= 0.0005$ in the proposed two-stage Bayesian FMS smoother. Figure 1 shows the test variable for uncertainty presence and absence declaration. As mentioned before, the test variable t_{i-d} formulated by the estimation error of the primary Bayesian FMS smoother $\hat{x}^p_{i-d}$ provides a reference value for determining the presence or absence of an uncertainty for the electric motor system. Figure 2 shows estimation errors for the second state indicating rotational speed for two smoothers, the primary FMS smoother with $M_p = 20$, the secondary FMS smoother with $M_s = 10$, and the existing two-stage FMS filter. According to the declaration rule (19) using the test variable (18) and the threshold value $\gamma = 7.88$ corresponding to PFA $= 0.0005$, the proposed two-stage Bayesian FMS smoother with two kinds of measurement windows provides the state estimate as shown in Figure 2 which show comparisons with three other estimation approaches. As shown in simulation results, the proposed two-stage Bayesian FMS smoother with two kinds of measurement windows can be better than the primary FMS smoother and the existing two-stage Bayesian FMS filter in terms of error magnitude and error convergence on the interval where modeling uncertainty exists. In addition, the proposed two-stage Bayesian FMS smoother can be better than the secondary FMS smoother when there is no temporary model uncertainty or after temporary model uncertainty is gone. These observations on computer simulations show that the proposed two-stage Bayesian FMS smoother can work well in temporarily uncertain systems as well as in certain systems.

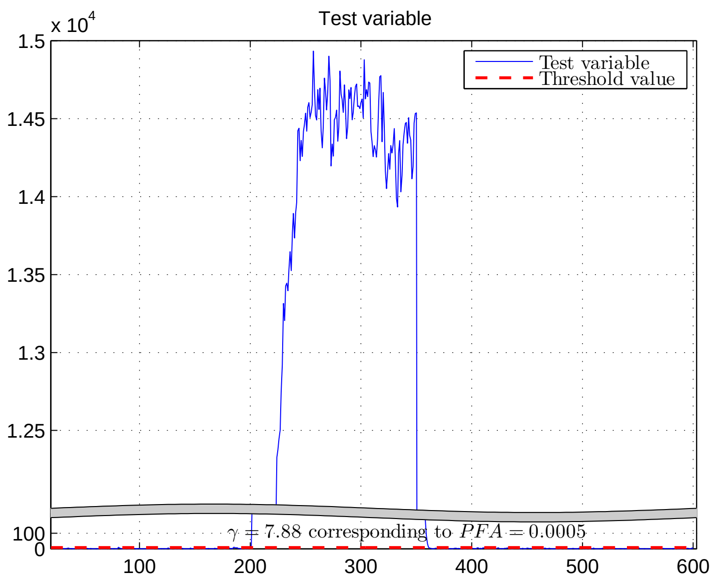

Figure 1. Test variable for the electric motor system

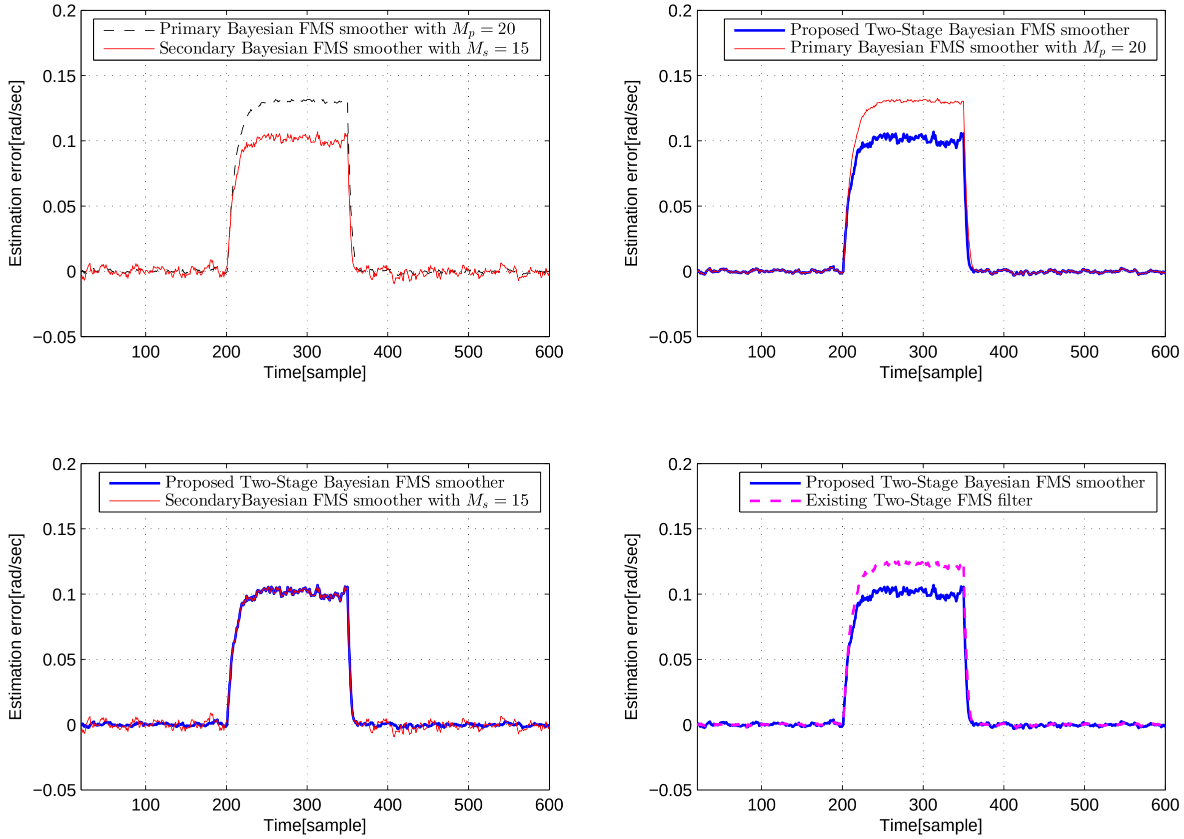

FIGURE 2. Estimation errors

5. **Conclusions.** This paper has proposed two-stage Bayesian FMS smoother with two kinds of measurement windows using the chi-square test statistic in order to cover the nominal system as well as the temporarily uncertain system. The simple matrix form for the Bayesian FMS smoother has been developed from the conditional density of the current state given finite past measurements. Then, one of the two smoothers, the primary Bayesian FMS smoother and the secondary Bayesian FMS smoother, with different measurement windows has been operated selectively to obtain the valid estimate according to the presence or absence of uncertainty. The primary Bayesian FMS smoother has been selected for the nominal system and the secondary Bayesian FMS smoother has been selected for the temporarily uncertain system, respectively. A declaration rule has been defined to indicate the presence or absence of uncertainty, operate the suitable one from two smoothers, and then obtain the valid smoothing estimate. The test variables for the declaration rule have been defined using the chi-squared distribution with one degree of freedom. Finally, extensive computer simulations have been performed for an electric motor system to verify the proposed two-stage Bayesian FMS smoother with two kinds of measurement windows and compare with the standard FMS smoother with one measurement window as well as the existing two-stage FMS filter. Through simulation results, it has been confirmed that the proposed two-stage Bayesian FMS smoother works well for both nominal systems and temporarily uncertain systems. It has been also shown that the proposed two-stage Bayesian FMS smoother can be remarkably better than the existing two-stage FMS filter for the temporarily uncertain system.

Since the research work for the FMS smoother is relatively inactive in the case where noises are nonzero-mean Gaussian, an alternative smoother for nonzero-mean noises can be researched as future work.

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