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IMPROVED CRITERION FOR STABILITY OF 2-D DISCRETE SYSTEMS INVOLVING SATURATION NONLINEARITIES AND VARIABLE DELAYS

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Abstract. *This paper investigates two-dimensional (2-D) discrete systems described by Fornasini Marchesini Second Local State Space (FMSLSS) model. The system is considered to be under the influence of delays and saturation overflow nonlinear effects. The stability criterion is in the form of linear matrix inequalities (LMIs). The Wirtinger-based inequality and reciprocal covexity approach is used to develop the criterion. Using numerical example, comparison of the proposed criterion is done with previously reported criterion to demonstrate the effectiveness of the criterion.*
Keywords: Discrete-time systems, Finite wordlength effects, Linear matrix inequality, Lyapunov stability, Delayed systems

1. Introduction.

The stability of two-dimensional (2-D) discrete systems under the combined influence of saturation nonlinearities and delays is an important problem [1, 2, 3, 4]. It is well known that the presence of delays and nonlinearities may lead to instabilities in the system. Delayed systems have been widely studied [1, 2, 3, 4, 5, 6].

The system described by Fornasini Marchesini Second Local State Space (FMSLSS) model under the influence of saturation nonlinearities and interval-like time varying delays was studied in [1]. The 2-D systems with constant delays and saturation nonlinearities were dealt in [2, 3]. In [4] the system under the influence of saturation nonlinearities, time varying delays and uncertainties was studied. However, there is further scope for reducing conservativeness of 2-D discrete systems in the presence of variable delays and saturation nonlinearities.

In this paper, we establish new stability criterion for two-dimensional discrete systems involving variable delays and saturation nonlinearities. Substantial contributions of the paper are: (i) A delay dependent stability criterion for 2-D discrete systems involving variable delays and saturation overflow nonlinear effects is established; (ii) Comparison is done with the recently reported works [1, 4]; (iii) An analytical exemplification illustrates the effectiveness of the developed criterion. The organization of this paper is as follows. In Section 2, the description of system under consideration is presented. Section 3 establishes the vital results of the paper. In Section 4, a numerical example is discussed emphasizing the suitability of the given approach. Finally, Section 5 concludes the paper.

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2. **System Description.** Throughout this paper, the standard notations used are: $\mathbb{R}^p$ represents the p-dimensional Euclidean space; $\mathbb{R}^{p\times q}$ denotes the set of $p \times q$ real matrices; $\mathbf{0}$ is the null vector or null matrix of proper dimension; $\boldsymbol{I}$ represents unit matrix of suitable dimension; $\boldsymbol{B}^T$ is the transpose of the matrix (or vector) $\boldsymbol{B}$; $\boldsymbol{B} > \mathbf{0}$ ($\geq \mathbf{0}$) denotes that $\boldsymbol{B}$ is a positive definite (semidefinite) symmetric matrix; $\boldsymbol{f}(\cdot)$ characterizes saturation nonlinearities; the symmetric terms in a symmetric matrix is symbolized by '$*$'; $\mathbb{Z}_+$ denotes a set of nonnegative integers; $col\{\cdot\}$ denotes a column matrix; for any square matrix, $sym\{\boldsymbol{B}\}$ stands for $\left\{\boldsymbol{B} + \boldsymbol{B}^T\right\}$ and $\boldsymbol{e}_i$ ($i = 1, 2, \ldots, r$) are block entry matrices, For example, $\boldsymbol{e}_2 = [\mathbf{0} \;\; \boldsymbol{I} \;\; \mathbf{0} \;\; \cdots \;\; \mathbf{0}]$.

Consider a category of 2-D discrete systems involving variable delays and saturation nonlinear effects modeled using Fornasini Marchesini Second Local State Space (FMSLSS) model, described as follows:

$$\boldsymbol{x}(\mu+1,\nu+1) = \boldsymbol{f}(\boldsymbol{y}(\mu,\nu)) = [f_1(y_1(\mu,\nu)) \quad f_2(y_2(\mu,\nu)) \quad \cdots \quad f_n(y_n(\mu,\nu))]^T \tag{1a}$$

$$\begin{aligned}\boldsymbol{y}(\mu,\nu) = {} & \boldsymbol{A}_1\boldsymbol{x}(\mu,\nu+1) + \boldsymbol{A}_2\boldsymbol{x}(\mu+1,\nu) + \boldsymbol{A}_{d_1}\boldsymbol{x}(\mu-\alpha_\mu,\nu+1) \\ & + \boldsymbol{A}_{d_2}\boldsymbol{x}(\mu+1,\nu-\beta_\nu)\end{aligned} \tag{1b}$$

where $\mu \in \mathbb{Z}_+$ and $\nu \in \mathbb{Z}_+$ are spatial coordinates; $\boldsymbol{x}(\mu,\nu) \in \mathbb{R}^n$ represents the local state vector; $\boldsymbol{A}_1 = \left[a^{I}_{\mu\nu}\right]$, $\boldsymbol{A}_2 = \left[a^{II}_{\mu\nu}\right]$, $\boldsymbol{A}_{d_1} = \left[a^{III}_{\mu\nu}\right]$, $\boldsymbol{A}_{d_2} = \left[a^{IV}_{\mu\nu}\right]$ are the known $n \times n$ constant matrices; α_μ and β_ν are variable delays along horizontal and vertical direction, respectively. Let α_μ and β_ν satisfy

$$\alpha_l \leq \alpha_\mu \leq \alpha_h, \quad \beta_l \leq \beta_\nu \leq \beta_h \tag{1c}$$

where α_l and β_l are fixed positive integers denoting the lower-limit delay through horizontal and vertical directions, respectively; α_h and β_h are fixed nonnegative integers denoting the upper-limit delay through horizontal and vertical directions, respectively. Suitable values of α_l, β_l, α_h, β_h are employed in an iterative manner so that (8) hold and leading to a range of α_h and β_h such that the system is stable.

The saturation nonlinear effects are given by

$$f_k(y_k(\mu,\nu)) = \begin{cases} y_k(\mu,\nu), & |y_k(\mu,\nu)| \leq 1 \\ 1, & y_k(\mu,\nu) > 1 \\ -1, & y_k(\mu,\nu) < -1 \end{cases} \tag{2}$$

$k = 1, 2, \ldots, n$, are under consideration.

It is assumed [1, 8] that system (1), i.e., (1(a)-1(c)) has a limited array of boundary requirements, i.e., two non-negative integers K and L prevail such that

$$\begin{aligned} &\boldsymbol{x}(\mu,\nu) = \mathbf{0}, \quad \forall \mu \geq K, \quad \nu = -\beta_h, -\beta_h+1, \ldots, 0, \\ &\boldsymbol{x}(\mu,\nu) = \boldsymbol{p}_{\mu,\nu}, \quad \forall 0 \leq \mu < K, \quad \nu = -\beta_h, -\beta_h+1, \ldots, 0, \\ &\boldsymbol{x}(\mu,\nu) = \mathbf{0}, \quad \forall \nu \geq L, \quad \mu = -\alpha_h, -\alpha_h+1, \ldots, 0, \\ &\boldsymbol{x}(\mu,\nu) = \boldsymbol{q}_{\mu\nu}, \quad \forall 0 \leq \nu < L, \quad \mu = -\alpha_h, -\alpha_h+1, \ldots, 0, \\ &\boldsymbol{p}_{\mu,\nu} = \boldsymbol{q}_{\mu,\nu}, \quad \text{when } \mu = 0 \text{ and } \nu = 0 \end{aligned} \tag{3}$$

Define

$$r_\mu = \sum_{\nu=1}^{4n} |a_{\mu\nu}|, \quad \mu = 1, 2, \ldots, n \tag{4}$$

where $a_{\mu\nu}$ is the (μ,ν)th entry of $\boldsymbol{A} = [\boldsymbol{A}_1 \;\; \boldsymbol{A}_2 \;\; \boldsymbol{A}_{d_1} \;\; \boldsymbol{A}_{d_2}]$ and assume that the elements of matrix $\boldsymbol{A}$ satisfy

$$r_\mu > 1, \quad \mu = 1, 2, \ldots, m \tag{5}$$

$$r_\mu \leq 1, \quad \mu = m+1, m+2, \ldots, n \tag{6}$$

where m is an integer between 0 and n.

Now, we have the definition that standardizes the concept of asymptotical stability of 2-D discrete systems under consideration.

Definition 2.1. [12] *The 2-D discrete system with time-varying delays* (1)-(3) *is asymptotically stable if* $\lim_{r\to\infty} \boldsymbol{X}_r = \boldsymbol{0}$, *where* $\boldsymbol{X}_r = \sup\{||\boldsymbol{x}(i,j)|| : i+j=r, i,j \in \mathbb{Z}_+\}$.

3. **Main Results.** In this section we present the major findings of the paper.

Theorem 3.1. *Given positive integers* α_l, α_h, β_l, β_h *satisfying* $0 < \alpha_l < \alpha_h$ *and* $0 < \beta_l < \beta_h$, *the system represented by* (1)-(6) *is globally asymptotically stable if there exist positive-definite symmetric matrices*

$$\bar{\boldsymbol{P}} = \begin{bmatrix} \bar{\boldsymbol{P}}_1 & \bar{\boldsymbol{P}}_2 & \bar{\boldsymbol{P}}_3 \\ * & \bar{\boldsymbol{P}}_4 & \bar{\boldsymbol{P}}_5 \\ * & * & \bar{\boldsymbol{P}}_6 \end{bmatrix} \in \mathbb{R}^{3n\times 3n}, \quad \hat{\boldsymbol{P}} = \begin{bmatrix} \hat{\boldsymbol{P}}_1 & \hat{\boldsymbol{P}}_2 & \hat{\boldsymbol{P}}_3 \\ * & \hat{\boldsymbol{P}}_4 & \hat{\boldsymbol{P}}_5 \\ * & * & \hat{\boldsymbol{P}}_6 \end{bmatrix} \in \mathbb{R}^{3n\times 3n},$$

$\bar{\boldsymbol{Q}}_\mu, \hat{\boldsymbol{Q}}_\nu (\mu,\nu = 1,2,3) \in \mathbb{R}^{n\times n}$, $\bar{\boldsymbol{R}}_\mu, \hat{\boldsymbol{R}}_\nu (\mu,\nu = 1,2) \in \mathbb{R}^{n\times n}$ *and the matrices*

$$\begin{bmatrix} \bar{\boldsymbol{Y}}_{11} & \bar{\boldsymbol{Y}}_{12} \\ \bar{\boldsymbol{Y}}_{21} & \bar{\boldsymbol{Y}}_{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \quad \begin{bmatrix} \hat{\boldsymbol{Y}}_{11} & \hat{\boldsymbol{Y}}_{12} \\ \hat{\boldsymbol{Y}}_{21} & \hat{\boldsymbol{Y}}_{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

such that

$$\begin{bmatrix} \bar{\boldsymbol{R}}_2 & \boldsymbol{0} & \bar{\boldsymbol{Y}}_{11} & \bar{\boldsymbol{Y}}_{12} \\ * & 3\bar{\boldsymbol{R}}_2 & \bar{\boldsymbol{Y}}_{21} & \bar{\boldsymbol{Y}}_{22} \\ * & * & \bar{\boldsymbol{R}}_2 & \boldsymbol{0} \\ * & * & * & 3\bar{\boldsymbol{R}}_2 \end{bmatrix} \geq \boldsymbol{0}, \quad \begin{bmatrix} \hat{\boldsymbol{R}}_2 & \boldsymbol{0} & \hat{\boldsymbol{Y}}_{11} & \hat{\boldsymbol{Y}}_{12} \\ * & 3\hat{\boldsymbol{R}}_2 & \hat{\boldsymbol{Y}}_{21} & \hat{\boldsymbol{Y}}_{22} \\ * & * & \hat{\boldsymbol{R}}_2 & \boldsymbol{0} \\ * & * & * & 3\hat{\boldsymbol{R}}_2 \end{bmatrix} \geq \boldsymbol{0} \tag{7}$$

and the following Linear Matrix Inequalities (LMIs) hold simultaneously:

$$\Xi(\alpha_\mu = \alpha_l,\ \beta_\nu = \beta_l) < \boldsymbol{0} \tag{8a}$$
$$\Xi(\alpha_\mu = \alpha_l,\ \beta_\nu = \beta_h) < \boldsymbol{0} \tag{8b}$$
$$\Xi(\alpha_\mu = \alpha_h,\ \beta_\nu = \beta_l) < \boldsymbol{0} \tag{8c}$$
$$\Xi(\alpha_\mu = \alpha_h,\ \beta_\nu = \beta_h) < \boldsymbol{0} \tag{8d}$$

where

$$\Xi(\alpha_\mu, \beta_\nu) = sym\{\boldsymbol{\Lambda}\} + \sum_{k=1}^{15} \boldsymbol{e}_k^T \Xi_{k,k} \boldsymbol{e}_k, \tag{9}$$

$$\begin{aligned} \boldsymbol{\Lambda} = {} & \boldsymbol{e}_1^T \left[\frac{\left(\bar{\boldsymbol{P}}_3^T - \bar{\boldsymbol{P}}_2^T\right)}{2} - 2\bar{\boldsymbol{R}}_1 \right] \boldsymbol{e}_5 + \boldsymbol{e}_1^T \left[\frac{-\bar{\boldsymbol{P}}_3^T}{2} \right] \boldsymbol{e}_7 + \boldsymbol{e}_1^T \left[\frac{\alpha_l \left(\bar{\boldsymbol{P}}_4 - \bar{\boldsymbol{P}}_2^T\right)}{2} + 3\bar{\boldsymbol{R}}_1 \right] \boldsymbol{e}_9 \\ & + \boldsymbol{e}_1^T \left[\frac{(\alpha_\mu - \alpha_l)\left(\bar{\boldsymbol{P}}_5^T - \bar{\boldsymbol{P}}_3^T\right)}{2} \right] \boldsymbol{e}_{11} + \boldsymbol{e}_1^T \left[\frac{(\alpha_h - \alpha_\mu)\left(\bar{\boldsymbol{P}}_5^T - \bar{\boldsymbol{P}}_3^T\right)}{2} \right] \boldsymbol{e}_{13} \\ & + \boldsymbol{e}_1^T \left[\frac{\bar{\boldsymbol{P}}_2}{2} - \alpha_l^2 \bar{\boldsymbol{R}}_1 - \alpha_{hl}^2 \bar{\boldsymbol{R}}_2 + \boldsymbol{C}^T \boldsymbol{A}_1 \right] \boldsymbol{e}_{15} + \boldsymbol{e}_2^T \left[\frac{\left(\hat{\boldsymbol{P}}_3^T - \hat{\boldsymbol{P}}_2^T\right)}{2} - 2\hat{\boldsymbol{R}}_1 \right] \boldsymbol{e}_6 \\ & + \boldsymbol{e}_2^T \left[\frac{-\hat{\boldsymbol{P}}_3^T}{2} \right] e_8 + \boldsymbol{e}_2^T \left[\frac{\beta_l \left(\hat{\boldsymbol{P}}_4 - \hat{\boldsymbol{P}}_2^T\right)}{2} + 3\hat{\boldsymbol{R}}_1 \right] \boldsymbol{e}_{10} \end{aligned}$$

$$
\begin{aligned}
&+\boldsymbol{e}_2^T\left[\frac{(\beta_\nu-\beta_l)\left(\hat{\boldsymbol{P}}_5^T-\hat{\boldsymbol{P}}_3^T\right)}{2}\right]\boldsymbol{e}_{12}+\boldsymbol{e}_2^T\left[\frac{(\beta_h-\beta_\nu)\left(\hat{\boldsymbol{P}}_5^T-\hat{\boldsymbol{P}}_3^T\right)}{2}\right]\boldsymbol{e}_{14}\\
&+\boldsymbol{e}_2^T\left[\frac{\hat{\boldsymbol{P}}_2}{2}-\beta_l^2\hat{\boldsymbol{R}}_1-\beta_{hl}^2\hat{\boldsymbol{R}}_2+\boldsymbol{C}^T\boldsymbol{A}_2\right]\boldsymbol{e}_{15}\\
&+\boldsymbol{e}_3^T\left[-2\bar{\boldsymbol{R}}_2-\bar{\boldsymbol{Y}}_{11}-\bar{\boldsymbol{Y}}_{21}-\bar{\boldsymbol{Y}}_{12}-\bar{\boldsymbol{Y}}_{22}\right]\boldsymbol{e}_5\\
&+\boldsymbol{e}_3^T\left[-2\bar{\boldsymbol{R}}_2-\bar{\boldsymbol{Y}}_{11}^T+\bar{\boldsymbol{Y}}_{12}^T+\bar{\boldsymbol{Y}}_{21}^T-\bar{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_7\\
&+\boldsymbol{e}_3^T\left[3\bar{\boldsymbol{R}}_2+\bar{\boldsymbol{Y}}_{21}+\bar{\boldsymbol{Y}}_{22}\right]\boldsymbol{e}_{11}+\boldsymbol{e}_3^T\left[3\bar{\boldsymbol{R}}_2-\bar{\boldsymbol{Y}}_{12}^T+\bar{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_{13}+\boldsymbol{e}_3^T\left[\boldsymbol{C}^T\boldsymbol{A}_{d_1}\right]\boldsymbol{e}_{15}\\
&+\boldsymbol{e}_4^T\left[-2\hat{\boldsymbol{R}}_2-\hat{\boldsymbol{Y}}_{11}-\hat{\boldsymbol{Y}}_{21}-\hat{\boldsymbol{Y}}_{12}-\hat{\boldsymbol{Y}}_{22}\right]\boldsymbol{e}_6\\
&+\boldsymbol{e}_4^T\left[-2\hat{\boldsymbol{R}}_2-\hat{\boldsymbol{Y}}_{11}^T+\hat{\boldsymbol{Y}}_{12}^T+\hat{\boldsymbol{Y}}_{21}^T-\hat{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_8+\boldsymbol{e}_4^T\left[3\hat{\boldsymbol{R}}_2+\hat{\boldsymbol{Y}}_{21}+\hat{\boldsymbol{Y}}_{22}\right]\boldsymbol{e}_{12}\\
&+\boldsymbol{e}_4^T\left[3\hat{\boldsymbol{R}}_2-\hat{\boldsymbol{Y}}_{12}^T+\hat{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_{14}+\boldsymbol{e}_4^T\left[\boldsymbol{C}^T\boldsymbol{A}_{d_2}\right]\boldsymbol{e}_{15}+\boldsymbol{e}_5^T\left[\bar{\boldsymbol{Y}}_{11}^T-\bar{\boldsymbol{Y}}_{12}^T+\bar{\boldsymbol{Y}}_{21}^T-\bar{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_7\\
&+\boldsymbol{e}_5^T\left[\frac{\alpha_l\left(\bar{\boldsymbol{P}}_5-\bar{\boldsymbol{P}}_4\right)}{2}+3\bar{\boldsymbol{R}}_1\right]\boldsymbol{e}_9+\boldsymbol{e}_5^T\left[\frac{(\alpha_\mu-\alpha_l)\left(\bar{\boldsymbol{P}}_6-\bar{\boldsymbol{P}}_5^T\right)}{2}+3\bar{\boldsymbol{R}}_2\right]\boldsymbol{e}_{11}\\
&+\boldsymbol{e}_5^T\left[\frac{(\alpha_h-\alpha_\mu)\left(\bar{\boldsymbol{P}}_6-\bar{\boldsymbol{P}}_5^T\right)}{2}+\bar{\boldsymbol{Y}}_{12}^T+\bar{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_{13}+\boldsymbol{e}_5^T\left[\frac{\left(\bar{\boldsymbol{P}}_3-\bar{\boldsymbol{P}}_2\right)}{2}\right]\boldsymbol{e}_{15}\\
&+\boldsymbol{e}_6^T\left[\hat{\boldsymbol{Y}}_{11}^T-\hat{\boldsymbol{Y}}_{12}^T+\hat{\boldsymbol{Y}}_{21}^T-\hat{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_8+\boldsymbol{e}_6^T\left[\frac{\beta_l\left(\hat{\boldsymbol{P}}_5-\hat{\boldsymbol{P}}_4\right)}{2}+3\hat{\boldsymbol{R}}_1\right]\boldsymbol{e}_{10}\\
&+\boldsymbol{e}_6^T\left[\frac{(\beta_\nu-\beta_l)\left(\hat{\boldsymbol{P}}_6-\hat{\boldsymbol{P}}_5^T\right)}{2}+3\hat{\boldsymbol{R}}_2\right]\boldsymbol{e}_{12}\\
&+\boldsymbol{e}_6^T\left[\frac{(\beta_h-\beta_\nu)\left(\hat{\boldsymbol{P}}_6-\hat{\boldsymbol{P}}_5^T\right)}{2}+\hat{\boldsymbol{Y}}_{12}^T+\hat{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_{14}+\boldsymbol{e}_6^T\left[\frac{\left(\hat{\boldsymbol{P}}_3-\hat{\boldsymbol{P}}_2\right)}{2}\right]\boldsymbol{e}_{15}\\
&+\boldsymbol{e}_7^T\left[\frac{-\alpha_l\bar{\boldsymbol{P}}_5}{2}\right]\boldsymbol{e}_9+\boldsymbol{e}_7^T\left[\frac{-(\alpha_\mu-\alpha_l)\bar{\boldsymbol{P}}_6}{2}-\bar{\boldsymbol{Y}}_{21}+\bar{\boldsymbol{Y}}_{22}\right]\boldsymbol{e}_{11}\\
&+\boldsymbol{e}_7^T\left[\frac{-(\alpha_h-\alpha_\mu)\bar{\boldsymbol{P}}_6}{2}+3\bar{\boldsymbol{R}}_2\right]\boldsymbol{e}_{13}+\boldsymbol{e}_7^T\left[\frac{-\bar{\boldsymbol{P}}_3}{2}\right]\boldsymbol{e}_{15}+\boldsymbol{e}_8^T\left[\frac{-\beta_l\hat{\boldsymbol{P}}_5}{2}\right]e_{10}\\
&+\boldsymbol{e}_8^T\left[\frac{-(\beta_\nu-\beta_l)\hat{\boldsymbol{P}}_6}{2}-\hat{\boldsymbol{Y}}_{21}+\hat{\boldsymbol{Y}}_{22}\right]\boldsymbol{e}_{12}+\boldsymbol{e}_8^T\left[\frac{-(\beta_h-\beta_\nu)\hat{\boldsymbol{P}}_6}{2}+3\hat{\boldsymbol{R}}_2\right]\boldsymbol{e}_{14}\\
&+\boldsymbol{e}_8^T\left[\frac{-\hat{\boldsymbol{P}}_3}{2}\right]\boldsymbol{e}_{15}+\boldsymbol{e}_9^T\left[\frac{\alpha_l\bar{\boldsymbol{P}}_2}{2}\right]\boldsymbol{e}_{15}+\boldsymbol{e}_{10}^T\left[\frac{\beta_l\hat{\boldsymbol{P}}_2}{2}\right]\boldsymbol{e}_{15}+\boldsymbol{e}_{11}^T\left[-\bar{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_{13}\\
&+\boldsymbol{e}_{11}^T\left[\frac{(\alpha_\mu-\alpha_l)\bar{\boldsymbol{P}}_3}{2}\right]\boldsymbol{e}_{15}+\boldsymbol{e}_{12}^T\left[-\hat{\boldsymbol{Y}}_{22}^T\right]\boldsymbol{e}_{14}+\boldsymbol{e}_{12}^T\left[\frac{(\beta_\nu-\beta_l)\hat{\boldsymbol{P}}_3}{2}\right]\boldsymbol{e}_{15}\\
&+\boldsymbol{e}_{13}^T\left[\frac{(\alpha_h-\alpha_\mu)\bar{\boldsymbol{P}}_3}{2}\right]\boldsymbol{e}_{15}+\boldsymbol{e}_{14}^T\left[\frac{(\beta_h-\beta_\nu)\hat{\boldsymbol{P}}_3}{2}\right]\boldsymbol{e}_{15} \qquad (10)
\end{aligned}
$$

and

$$\boldsymbol{\Xi}_{1,1} = -\bar{\boldsymbol{P}}_1 + \frac{\left(\bar{\boldsymbol{P}}_2 + \bar{\boldsymbol{P}}_2^T\right)}{2} + \bar{\boldsymbol{Q}}_1 + \bar{\boldsymbol{Q}}_2 + \bar{\boldsymbol{Q}}_3 + \alpha_{hl}\bar{\boldsymbol{Q}}_3 + \alpha_l^2\bar{\boldsymbol{R}}_1 + \alpha_{hl}^2\bar{\boldsymbol{R}}_2 - 4\bar{\boldsymbol{R}}_1 \tag{11a}$$

$$\boldsymbol{\Xi}_{2,2} = -\hat{\boldsymbol{P}}_1 + \frac{\left(\hat{\boldsymbol{P}}_2 + \hat{\boldsymbol{P}}_2^T\right)}{2} + \hat{\boldsymbol{Q}}_1 + \hat{\boldsymbol{Q}}_2 + \hat{\boldsymbol{Q}}_3 + \beta_{hl}\hat{\boldsymbol{Q}}_3 + \beta_l^2\hat{\boldsymbol{R}}_1 + \beta_{hl}^2\hat{\boldsymbol{R}}_2 - 4\hat{\boldsymbol{R}}_1 \tag{11b}$$

$$\boldsymbol{\Xi}_{3,3} = -\bar{\boldsymbol{Q}}_3 - 8\bar{\boldsymbol{R}}_2 + \bar{\boldsymbol{Y}}_{11} + \bar{\boldsymbol{Y}}_{11}^T + \bar{\boldsymbol{Y}}_{12} + \bar{\boldsymbol{Y}}_{12}^T - \bar{\boldsymbol{Y}}_{21} - \bar{\boldsymbol{Y}}_{21}^T - \bar{\boldsymbol{Y}}_{22} - \bar{\boldsymbol{Y}}_{22}^T \tag{11c}$$

$$\boldsymbol{\Xi}_{4,4} = -\hat{\boldsymbol{Q}}_3 - 8\hat{\boldsymbol{R}}_2 + \hat{\boldsymbol{Y}}_{11} + \hat{\boldsymbol{Y}}_{11}^T + \hat{\boldsymbol{Y}}_{12} + \hat{\boldsymbol{Y}}_{12}^T - \hat{\boldsymbol{Y}}_{21} - \hat{\boldsymbol{Y}}_{21}^T - \hat{\boldsymbol{Y}}_{22} - \hat{\boldsymbol{Y}}_{22}^T \tag{11d}$$

$$\boldsymbol{\Xi}_{5,5} = -\bar{\boldsymbol{Q}}_1 - 4\left(\bar{\boldsymbol{R}}_1 + \bar{\boldsymbol{R}}_2\right) \tag{11e}$$

$$\boldsymbol{\Xi}_{6,6} = -\hat{\boldsymbol{Q}}_1 - 4\left(\hat{\boldsymbol{R}}_1 + \hat{\boldsymbol{R}}_2\right) \tag{11f}$$

$$\boldsymbol{\Xi}_{7,7} = -\bar{\boldsymbol{Q}}_2 - 4\bar{\boldsymbol{R}}_2 \tag{11g}$$

$$\boldsymbol{\Xi}_{8,8} = -\hat{\boldsymbol{Q}}_2 - 4\hat{\boldsymbol{R}}_2 \tag{11h}$$

$$\boldsymbol{\Xi}_{9,9} = -3\bar{\boldsymbol{R}}_1 \tag{11i}$$

$$\boldsymbol{\Xi}_{10,10} = -3\hat{\boldsymbol{R}}_1 \tag{11j}$$

$$\boldsymbol{\Xi}_{11,11} = \boldsymbol{\Xi}_{13,13} = -3\bar{\boldsymbol{R}}_2 \tag{11k}$$

$$\boldsymbol{\Xi}_{12,12} = \boldsymbol{\Xi}_{14,14} = -3\hat{\boldsymbol{R}}_2 \tag{11l}$$

$$\boldsymbol{\Xi}_{15,15} = \bar{\boldsymbol{P}}_1 + \alpha_l^2\bar{\boldsymbol{R}}_1 + \alpha_{hl}^2\bar{\boldsymbol{R}}_2 + \hat{\boldsymbol{P}}_1 + \beta_l^2\hat{\boldsymbol{R}}_1 + \beta_{hl}^2\hat{\boldsymbol{R}}_2 - \left(\boldsymbol{C} + \boldsymbol{C}^T\right) \tag{11m}$$

$$\alpha_{hl} = \alpha_h - \alpha_l, \quad \beta_{hl} = \beta_h - \beta_l \tag{11n}$$

also the matrix $\boldsymbol{C} = [c_{\mu\nu}] \in \boldsymbol{R}^{n\times n}$ *is characterized by*

$$c_{\mu\mu} = \sum_{\nu=1,\nu\neq\mu}^{n} (g_{\mu\nu} + h_{\mu\nu}), \quad \mu = 1, 2, \ldots, m \tag{12a}$$

$$c_{\mu\nu} = \begin{cases} g_{\mu\nu} - h_{\mu\nu}, & \mu, \nu = 1, 2, \ldots, m \quad (\mu \neq \nu) \\ \dfrac{g_{\mu\nu} - h_{\mu\nu}}{s_\nu}, & \mu = 1, 2, \ldots, m, \quad \nu = m+1, m+2, \ldots, n \end{cases} \tag{12b}$$

$$g_{\mu\nu} > 0, \quad h_{\mu\nu} > 0, \quad \mu = 1, 2, \ldots, m, \quad \nu = 1, 2, \ldots, n \quad (\mu \neq \nu) \tag{12c}$$

and the parameters $s_{m+1}, s_{m+2}, \ldots, s_n$ *in* (12b) *are defined by*

$$s_\mu = \sum_{\nu=1}^{m} \left\{\left|a_{\mu\nu}^{I}\right| + \left|a_{\mu\nu}^{II}\right| + \left|a_{\mu\nu}^{III}\right| + \left|a_{\mu\nu}^{IV}\right|\right\} + \sum_{\nu=m+1}^{n} p_\nu \left\{\left|a_{\mu\nu}^{I}\right| + \left|a_{\mu\nu}^{II}\right| + \left|a_{\mu\nu}^{III}\right| + \left|a_{\mu\nu}^{IV}\right|\right\}, \quad \mu = m+1, m+2, \ldots, n \tag{12d}$$

$$p_\mu = \sum_{\nu=1}^{4n} \left|a_{\mu\nu}\right|, \quad \mu = m+1, m+2, \ldots, n \tag{12e}$$

where $a_{\mu\nu}$ *is the* (μ,ν)*th entry of* $\boldsymbol{A} = [\boldsymbol{A}_1 \ \ \boldsymbol{A}_2 \ \ \boldsymbol{A}_{d_1} \ \ \boldsymbol{A}_{d_2}]$.

Proof: Consider the following 2-D Lyapunov-Krasovskii functional:

$$V(\boldsymbol{x}(\mu,\nu)) = \bar{V}(\boldsymbol{x}(\mu,\nu)) + \hat{V}(\boldsymbol{x}(\mu,\nu)) \tag{13}$$

where

$$\bar{V}(\boldsymbol{x}(\mu,\nu)) = \boldsymbol{\Gamma}_h^T(\mu,\nu)\bar{\boldsymbol{P}}\boldsymbol{\Gamma}_h(\mu,\nu) + \sum_{r=-\alpha_l}^{-1} \boldsymbol{x}^T(\mu+r,\nu)\bar{\boldsymbol{Q}}_1\boldsymbol{x}(\mu+r,\nu)$$

$$
\begin{aligned}
&+ \sum_{r=-\alpha_h}^{-1} \boldsymbol{x}^T(\mu+r,\nu)\bar{\boldsymbol{Q}}_2\boldsymbol{x}(\mu+r,\nu) \\
&+ \sum_{\theta=-\alpha_h}^{-\alpha_l}\sum_{r=\theta}^{-1} \boldsymbol{x}^T(\mu+r,\nu)\bar{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+r,\nu) \\
&+ \alpha_l \sum_{\theta=-\alpha_l+1}^{0}\sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_h^T(\mu+r,\nu)\bar{\boldsymbol{R}}_1\boldsymbol{\eta}_h(\mu+r,\nu) \\
&+ \alpha_{hl} \sum_{\theta=-\alpha_h+1}^{-\alpha_l}\sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_h^T(\mu+r,\nu)\bar{\boldsymbol{R}}_2\boldsymbol{\eta}_h(\mu+r,\nu)
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\hat{V}(\boldsymbol{x}(\mu,\nu)) = {} & \boldsymbol{\Gamma}_v^T(\mu,\nu)\hat{\boldsymbol{P}}\boldsymbol{\Gamma}_v(\mu,\nu) + \sum_{r=-\beta_l}^{-1} \boldsymbol{x}^T(\mu,\nu+r)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(\mu,\nu+r) \\
&+ \sum_{r=-\beta_h}^{-1} \boldsymbol{x}^T(\mu,\nu+r)\hat{\boldsymbol{Q}}_2\boldsymbol{x}(\mu,\nu+r) \\
&+ \sum_{\theta=-\beta_h}^{-\beta_l}\sum_{r=\theta}^{-1} \boldsymbol{x}^T(\mu,\nu+r)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(\mu,\nu+r) \\
&+ \beta_l \sum_{\theta=-\beta_l+1}^{0}\sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_v^T(\mu,\nu+r)\hat{\boldsymbol{R}}_1\boldsymbol{\eta}_v(\mu,\nu+r) \\
&+ \beta_{hl} \sum_{\theta=-\beta_h+1}^{-\beta_l}\sum_{r=-1+\theta}^{-1} \boldsymbol{\eta}_v^T(\mu,\nu+r)\hat{\boldsymbol{R}}_2\boldsymbol{\eta}_v(\mu,\nu+r)
\end{aligned} \tag{15}
$$

and

$$
\boldsymbol{\Gamma}_h^T(\mu,\nu) = \left[\boldsymbol{x}^T(\mu,\nu) \ \sum_{r=-\alpha_l}^{-1} \boldsymbol{x}^T(\mu+r,\nu) \ \sum_{r=-\alpha_h}^{-\alpha_l-1} \boldsymbol{x}^T(\mu+r,\nu)\right] \tag{16a}
$$

$$
\boldsymbol{\Gamma}_v^T(\mu,\nu) = \left[\boldsymbol{x}^T(\mu,\nu) \ \sum_{r=-\beta_l}^{-1} \boldsymbol{x}^T(\mu,\nu+r) \ \sum_{r=-\beta_h}^{-\beta_l-1} \boldsymbol{x}^T(\mu,\nu+r)\right] \tag{16b}
$$

$$
\boldsymbol{\eta}_h(\mu,\nu+1) = \boldsymbol{x}(\mu+1,\nu+1) - \boldsymbol{x}(\mu,\nu+1) = \boldsymbol{f}(\boldsymbol{y}(\mu,\nu)) - \boldsymbol{x}(\mu,\nu+1) \tag{16c}
$$

$$
\boldsymbol{\eta}_v(\mu+1,\nu) = \boldsymbol{x}(\mu+1,\nu+1) - \boldsymbol{x}(\mu+1,\nu) = \boldsymbol{f}(\boldsymbol{y}(\mu,\nu)) - \boldsymbol{x}(\mu+1,\nu) \tag{16d}
$$

The finite difference of Lyapunov functional along the trajectories of system (1) is given by

$$
\Delta V(\boldsymbol{x}(\mu,\nu)) = \Delta_1\bar{V}(\boldsymbol{x}(\mu,\nu)) + \Delta_2\hat{V}(\boldsymbol{x}(\mu,\nu)) \tag{17}
$$

where

$$
\begin{aligned}
\Delta_1\bar{V}(\boldsymbol{x}(\mu,\nu)) = {} & \bar{V}(\boldsymbol{x}(\mu+1,\nu+1)) - \bar{V}(\boldsymbol{x}(\mu,\nu+1)) \\
= {} & \bar{\boldsymbol{\xi}}^T(\mu,\nu)\bar{\boldsymbol{\phi}}(\mu,\nu)\bar{\boldsymbol{\xi}}(\mu,\nu) + \boldsymbol{x}^T(\mu,\nu+1)\bar{\boldsymbol{Q}}_1\boldsymbol{x}(\mu,\nu+1) \\
&- \boldsymbol{x}^T(\mu-\alpha_l,\nu+1)\bar{\boldsymbol{Q}}_1\boldsymbol{x}(\mu-\alpha_l,\nu+1) + \boldsymbol{x}^T(\mu,\nu+1)\bar{\boldsymbol{Q}}_2\boldsymbol{x}(\mu,\nu+1) \\
&- \boldsymbol{x}^T(\mu-\alpha_h,\nu+1)\bar{\boldsymbol{Q}}_2\boldsymbol{x}(\mu-\alpha_h,\nu+1) + \boldsymbol{x}^T(\mu,\nu+1)\bar{\boldsymbol{Q}}_3\boldsymbol{x}(\mu,\nu+1) \\
&+ \alpha_{hl}\boldsymbol{x}^T(\mu,\nu+1)\bar{\boldsymbol{Q}}_3\boldsymbol{x}(\mu,\nu+1) \\
&- \sum_{r=-\alpha_h}^{-\alpha_l} \boldsymbol{x}^T(\mu+r,\nu+1)\bar{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+r,\nu+1)
\end{aligned}
$$

$$\begin{aligned}
&+\alpha_l^2\boldsymbol{\eta}_h^T(\mu,\nu+1)\bar{\boldsymbol{R}}_1\boldsymbol{\eta}_h(\mu,\nu+1)\\
&-\alpha_l\sum_{r=-\alpha_l}^{-1}\boldsymbol{\eta}_h^T(\mu+r,\nu+1)\bar{\boldsymbol{R}}_1\boldsymbol{\eta}_h(\mu+r,\nu+1)\\
&+\alpha_{hl}^2\boldsymbol{\eta}_h^T(\mu,\nu+1)\bar{\boldsymbol{R}}_2\boldsymbol{\eta}_h(\mu,\nu+1)\\
&-\alpha_{hl}\sum_{r=-\alpha_h}^{-\alpha_l-1}\boldsymbol{\eta}_h^T(\mu+r,\nu+1)\bar{\boldsymbol{R}}_2\boldsymbol{\eta}_h(\mu+r,\nu+1)
\end{aligned}\tag{18}$$

$$\begin{aligned}
\Delta_2\hat{V}(\boldsymbol{x}(\mu,\nu)) =\ & \hat{V}(\boldsymbol{x}(\mu+1,\nu+1))-\hat{V}(\boldsymbol{x}(\mu+1,\nu))\\
=\ & \bar{\boldsymbol{\xi}}^T(\mu,\nu)\hat{\boldsymbol{\phi}}(\mu,\nu)\bar{\boldsymbol{\xi}}(\mu,\nu)+\boldsymbol{x}^T(\mu+1,\nu)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(\mu+1,\nu)\\
&-\boldsymbol{x}^T(\mu+1,\nu-\beta_l)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(\mu+1,\nu-\beta_l)+\boldsymbol{x}^T(\mu+1,\nu)\hat{\boldsymbol{Q}}_2\boldsymbol{x}(\mu+1,\nu)\\
&-\boldsymbol{x}^T(\mu+1,\nu-\beta_h)\hat{\boldsymbol{Q}}_2\boldsymbol{x}(\mu+1,\nu-\beta_h)+\boldsymbol{x}^T(\mu+1,\nu)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+1,\nu)\\
&+\beta_{hl}\boldsymbol{x}^T(\mu+1,\nu)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+1,\nu)\\
&-\sum_{r=-\beta_h}^{-\beta_l}\boldsymbol{x}^T(\mu+1,\nu+r)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+1,\nu+r)\\
&+\beta_l^2\boldsymbol{\eta}_v^T(\mu+1,\nu)\hat{\boldsymbol{R}}_1\boldsymbol{\eta}_v(\mu+1,\nu)\\
&-\beta_l\sum_{r=-\beta_l}^{-1}\boldsymbol{\eta}_v^T(\mu+1,\nu+r)\hat{\boldsymbol{R}}_1\boldsymbol{\eta}_v(\mu+1,\nu+r)\\
&+\beta_{hl}^2\boldsymbol{\eta}_v^T(\mu+1,\nu)\hat{\boldsymbol{R}}_2\boldsymbol{\eta}_v(\mu+1,\nu)\\
&-\beta_{hl}\sum_{r=-\beta_h}^{-\beta_l-1}\boldsymbol{\eta}_v^T(\mu+1,\nu+r)\hat{\boldsymbol{R}}_2\boldsymbol{\eta}_v(\mu+1,\nu+r)
\end{aligned}\tag{19}$$

$$\bar{\boldsymbol{\xi}}^T(\mu,\nu)=col\left\{\begin{bmatrix}\boldsymbol{x}^T(\mu,\nu+1)\\ \boldsymbol{x}^T(\mu+1,\nu)\\ \boldsymbol{x}^T(\mu-\alpha_\mu,\nu+1)\\ \boldsymbol{x}^T(\mu+1,\nu-\beta_\nu)\\ \boldsymbol{x}^T(\mu-\alpha_l,\nu+1)\end{bmatrix},\begin{bmatrix}\boldsymbol{x}^T(\mu+1,\nu-\beta_l)\\ \boldsymbol{x}^T(\mu-\alpha_h,\nu+1)\\ \boldsymbol{x}^T(\mu+1,\nu-\beta_h)\\ \boldsymbol{\chi}^T(\mu,0,\alpha_l)\\ \boldsymbol{\chi}^T(\mu,0,\beta_l)\end{bmatrix},\begin{bmatrix}\boldsymbol{\chi}^T(\mu,\alpha_l,\alpha_\mu)\\ \boldsymbol{\chi}^T(\nu,\beta_l,\beta_\nu)\\ \boldsymbol{\chi}^T(\mu,\alpha_\mu,\alpha_h)\\ \boldsymbol{\chi}^T(\nu,\beta_\nu,\beta_h)\end{bmatrix}\right\}\tag{20}$$

$$\begin{aligned}
\bar{\boldsymbol{\phi}}(\mu,\nu)=\ &\boldsymbol{e}_1^T\left[-\bar{\boldsymbol{P}}_1+\frac{\left(\bar{\boldsymbol{P}}_2+\bar{\boldsymbol{P}}_2^T\right)}{2}\right]\boldsymbol{e}_1+sym\left\{\boldsymbol{e}_1^T\left[\frac{\left(\bar{\boldsymbol{P}}_3^T-\bar{\boldsymbol{P}}_2^T\right)}{2}\right]\boldsymbol{e}_5+\boldsymbol{e}_1^T\left[\frac{-\bar{\boldsymbol{P}}_3^T}{2}\right]\boldsymbol{e}_7\right.\\
&+\boldsymbol{e}_1^T\left[\frac{\alpha_l\left(\bar{\boldsymbol{P}}_4-\bar{\boldsymbol{P}}_2^T\right)}{2}\right]\boldsymbol{e}_9+\boldsymbol{e}_1^T\left[\frac{(\alpha_\mu-\alpha_l)\left(\bar{\boldsymbol{P}}_5^T-\bar{\boldsymbol{P}}_3^T\right)}{2}\right]\boldsymbol{e}_{11}\\
&+\boldsymbol{e}_1^T\left[\frac{(\alpha_h-\alpha_\mu)\left(\bar{\boldsymbol{P}}_5^T-\bar{\boldsymbol{P}}_3^T\right)}{2}\right]\boldsymbol{e}_{13}+\boldsymbol{e}_5^T\left[\frac{\alpha_l\left(\bar{\boldsymbol{P}}_5-\bar{\boldsymbol{P}}_4\right)}{2}\right]\boldsymbol{e}_9\\
&+\boldsymbol{e}_5^T\left[\frac{(\alpha_\mu-\alpha_l)\left(\bar{\boldsymbol{P}}_6-\bar{\boldsymbol{P}}_5^T\right)}{2}\right]\boldsymbol{e}_{11}+\boldsymbol{e}_5^T\left[\frac{(\alpha_h-\alpha_\mu)\left(\bar{\boldsymbol{P}}_6-\bar{\boldsymbol{P}}_5^T\right)}{2}\right]\boldsymbol{e}_{13}\\
&+\boldsymbol{e}_7^T\left[\frac{-\alpha_l\bar{\boldsymbol{P}}_5}{2}\right]\boldsymbol{e}_9+\boldsymbol{e}_7^T\left[\frac{-(\alpha_\mu-\alpha_l)\bar{\boldsymbol{P}}_6}{2}\right]\boldsymbol{e}_{11}
\end{aligned}$$

$$+ \boldsymbol{e}_7^T \left[\frac{-(\alpha_h - \alpha_\mu)\bar{\boldsymbol{P}}_6}{2}\right] \boldsymbol{e}_{13} \Bigg\} \tag{21}$$

$$\hat{\boldsymbol{\phi}}(\mu,\nu) = \boldsymbol{e}_2^T \left[-\hat{\boldsymbol{P}}_1 + \frac{\left(\hat{\boldsymbol{P}}_2 + \hat{\boldsymbol{P}}_2^T\right)}{2}\right] \boldsymbol{e}_2 + sym \Bigg\{ \boldsymbol{e}_2^T \left[\frac{\left(\hat{\boldsymbol{P}}_3^T - \hat{\boldsymbol{P}}_2^T\right)}{2}\right] \boldsymbol{e}_6 + \boldsymbol{e}_2^T \left[\frac{-\hat{\boldsymbol{P}}_3^T}{2}\right] \boldsymbol{e}_8$$
$$+ \boldsymbol{e}_2^T \left[\frac{\beta_l \left(\hat{\boldsymbol{P}}_4 - \hat{\boldsymbol{P}}_2^T\right)}{2}\right] \boldsymbol{e}_{10} + \boldsymbol{e}_2^T \left[\frac{(\beta_\nu - \beta_l)\left(\hat{\boldsymbol{P}}_5^T - \hat{\boldsymbol{P}}_3^T\right)}{2}\right] \boldsymbol{e}_{12}$$
$$+ \boldsymbol{e}_2^T \left[\frac{(\beta_h - \beta_\nu)\left(\hat{\boldsymbol{P}}_5^T - \hat{\boldsymbol{P}}_3^T\right)}{2}\right] \boldsymbol{e}_{14} + \boldsymbol{e}_6^T \left[\frac{\beta_l \left(\hat{\boldsymbol{P}}_5 - \hat{\boldsymbol{P}}_4\right)}{2}\right] \boldsymbol{e}_{10}$$
$$+ \boldsymbol{e}_6^T \left[\frac{(\beta_\nu - \beta_l)\left(\hat{\boldsymbol{P}}_6 - \hat{\boldsymbol{P}}_5^T\right)}{2}\right] \boldsymbol{e}_{12} + \boldsymbol{e}_6^T \left[\frac{(\beta_h - \beta_\nu)\left(\hat{\boldsymbol{P}}_6 - \hat{\boldsymbol{P}}_5^T\right)}{2}\right] \boldsymbol{e}_{14}$$
$$+ \boldsymbol{e}_8^T \left[\frac{-\beta_l \hat{\boldsymbol{P}}_5}{2}\right] \boldsymbol{e}_{10} + \boldsymbol{e}_8^T \left[\frac{-(\beta_\nu - \beta_l)\hat{\boldsymbol{P}}_6}{2}\right] \boldsymbol{e}_{12}$$
$$+ \boldsymbol{e}_8^T \left[\frac{-(\beta_h - \beta_\nu)\hat{\boldsymbol{P}}_6}{2}\right] \boldsymbol{e}_{14} \Bigg\} \tag{22}$$

and the terms $\boldsymbol{\chi}^T(\cdot)$ in (20) are obtained from Wirtinger inequality defined in [9].

Note that

$$-\sum_{r=-\alpha_h}^{-\alpha_l} \boldsymbol{x}^T(\mu+r,\nu+1)\bar{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+r,\nu+1) \le -\boldsymbol{x}^T(\mu-\alpha_\mu,\nu+1)\bar{\boldsymbol{Q}}_3\boldsymbol{x}(\mu-\alpha_\mu,\nu+1) \tag{23}$$

and

$$-\sum_{r=-\beta_h}^{-\beta_l} \boldsymbol{x}^T(\mu+1,\nu+r)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+1,\nu+r) \le -\boldsymbol{x}^T(\mu+1,\nu-\beta_\nu)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(\mu+1,\nu-\beta_\nu) \tag{24}$$

Next, by employing Discrete Wirtiger-based inequality [9], we obtain the following expressions

$$-\alpha_l \sum_{r=-\alpha_l}^{-1} \boldsymbol{\eta}_h^T(\mu+r,\nu+1)\bar{\boldsymbol{R}}_1\boldsymbol{\eta}_h(\mu+r,\nu+1)$$
$$\le -\begin{bmatrix} \boldsymbol{x}(\mu,\nu+1)-\boldsymbol{x}(\mu-\alpha_l,\nu+1) \\ \boldsymbol{x}(\mu,\nu+1)+\boldsymbol{x}(\mu-\alpha_l,\nu+1)-\boldsymbol{\chi}(\mu,0,\alpha_l) \end{bmatrix}^T \begin{bmatrix} \bar{\boldsymbol{R}}_1 & \boldsymbol{0} \\ \boldsymbol{0} & 3\bar{\boldsymbol{R}}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\mu,\nu+1)-\boldsymbol{x}(\mu-\alpha_l,\nu+1) \\ \boldsymbol{x}(\mu,\nu+1)+\boldsymbol{x}(\mu-\alpha_l,\nu+1)-\boldsymbol{\chi}(\mu,0,\alpha_l) \end{bmatrix} \tag{25}$$

and

$$-\beta_l \sum_{r=-\beta_l}^{-1} \boldsymbol{\eta}_v^T(\mu+1,\nu+r)\hat{\boldsymbol{R}}_1\boldsymbol{\eta}_v(\mu+1,\nu+r)$$
$$\le -\begin{bmatrix} \boldsymbol{x}(\mu+1,\nu)-\boldsymbol{x}(\mu+1,\nu-\beta_l) \\ \boldsymbol{x}(\mu+1,\nu)+\boldsymbol{x}(\mu+1,\nu-\beta_l)-\boldsymbol{\chi}(\nu,0,\beta_l) \end{bmatrix}^T \begin{bmatrix} \hat{\boldsymbol{R}}_1 & \boldsymbol{0} \\ \boldsymbol{0} & 3\hat{\boldsymbol{R}}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(\mu+1,\nu)-\boldsymbol{x}(\mu+1,\nu-\beta_l) \\ \boldsymbol{x}(\mu+1,\nu)+\boldsymbol{x}(\mu+1,\nu-\beta_l)-\boldsymbol{\chi}(\nu,0,\beta_l) \end{bmatrix} \tag{26}$$

Now, let us define

$$\bar{\gamma}_1 = \boldsymbol{x}(\mu-\alpha_l,\nu+1) - \boldsymbol{x}(\mu-\alpha_\mu,\nu+1) \tag{27a}$$
$$\bar{\gamma}_2 = \boldsymbol{x}(\mu-\alpha_l,\nu+1) + \boldsymbol{x}(\mu-\alpha_\mu,\nu+1) - \boldsymbol{\chi}(\mu,\alpha_l,\alpha_\mu) \tag{27b}$$

$$\bar{\gamma}_3 = \boldsymbol{x}(\mu - \alpha_\mu, \nu + 1) - \boldsymbol{x}(\mu - \alpha_h, \nu + 1) \tag{27c}$$
$$\bar{\gamma}_4 = \boldsymbol{x}(\mu - \alpha_\mu, \nu + 1) + \boldsymbol{x}(\mu - \alpha_h, \nu + 1) - \boldsymbol{\chi}(\mu, \alpha_\mu, \alpha_h) \tag{27d}$$
$$\hat{\gamma}_1 = \boldsymbol{x}(\mu + 1, \nu - \beta_l) - \boldsymbol{x}(\mu + 1, \nu - \beta_\nu) \tag{27e}$$
$$\hat{\gamma}_2 = \boldsymbol{x}(\mu + 1, \nu - \beta_l) + \boldsymbol{x}(\mu + 1, \nu - \beta_\nu) - \boldsymbol{\chi}(\nu, \beta_l, \beta_\nu) \tag{27f}$$
$$\hat{\gamma}_3 = \boldsymbol{x}(\mu + 1, \nu - \beta_\nu) - \boldsymbol{x}(\mu + 1, \nu - \beta_h) \tag{27g}$$
$$\hat{\gamma}_4 = \boldsymbol{x}(\mu + 1, \nu - \beta_\nu) + \boldsymbol{x}(\mu + 1, \nu - \beta_h) - \boldsymbol{\chi}(\nu, \beta_\nu, \beta_h) \tag{27h}$$

Let the quantity 'δ' be expressed as

$$\begin{aligned}\delta = &\sum_{k=1}^{m}[y_k(\mu,\nu) - f_k(y_k(\mu,\nu))]\left[c_{kk}f_k(y_k(\mu,\nu)) + \sum_{l=1,k\neq l}^{m} c_{kl}f_l(y_l(\mu,\nu))\right.\\ &\left. + \sum_{l=m+1,k\neq l}^{n} c_{kl}s_l\frac{f_l(y_l(\mu,\nu))}{s_l}\right] + \sum_{k=m+1}^{n}[y_k(\mu,\nu) - f_k(y_k(\mu,\nu))]\left[c_{kk}f_k(y_k(\mu,\nu))\right.\\ &\left. + \sum_{l=1,k\neq l}^{n} c_{kl}f_l(y_l(\mu,\nu))\right]\\ = &\ \boldsymbol{y}^T(\mu,\nu)\boldsymbol{C}\boldsymbol{f}(\boldsymbol{y}(\mu,\nu)) + \boldsymbol{f}^T(\boldsymbol{y}(\mu,\nu))\boldsymbol{C}^T\boldsymbol{f}(\boldsymbol{y}(\mu,\nu))\\ &- \boldsymbol{f}^T(\boldsymbol{y}(\mu,\nu))\left(\boldsymbol{C} + \boldsymbol{C}^T\right)\boldsymbol{f}(\boldsymbol{y}(\mu,\nu))\end{aligned} \tag{28}$$

By applying reciprocal convexity method [10, 11] and using (28), the following relation is established

$$\Delta V(\boldsymbol{x}(\mu,\nu)) \leq \boldsymbol{\xi}^T(\mu,\nu)\Xi(\alpha_\mu,\beta_\nu)\boldsymbol{\xi}(\mu,\nu) - \delta \tag{29}$$

where

$$\boldsymbol{\xi}^T(\mu,\nu) = col\left\{\begin{bmatrix}\boldsymbol{x}^T(\mu,\nu+1)\\ \boldsymbol{x}^T(\mu+1,\nu)\\ \boldsymbol{x}^T(\mu-\alpha_\mu,\nu+1)\\ \boldsymbol{x}^T(\mu+1,\nu-\beta_\nu)\\ \boldsymbol{x}^T(\mu-\alpha_l,\nu+1)\end{bmatrix}, \begin{bmatrix}\boldsymbol{x}^T(\mu+1,\nu-\beta_l)\\ \boldsymbol{x}^T(\mu-\alpha_h,\nu+1)\\ \boldsymbol{x}^T(\mu+1,\nu-\beta_h)\\ \boldsymbol{\chi}^T(\mu,0,\alpha_l)\\ \boldsymbol{\chi}^T(\nu,0,\beta_l)\end{bmatrix}, \begin{bmatrix}\boldsymbol{\chi}^T(\mu,\alpha_l,\alpha_\mu)\\ \boldsymbol{\chi}^T(\nu,\beta_l,\beta_\nu)\\ \boldsymbol{\chi}^T(\mu,\alpha_\mu,\alpha_h)\\ \boldsymbol{\chi}^T(\nu,\beta_\nu,\beta_h)\\ \boldsymbol{f}^T(\boldsymbol{y}(\mu,\nu))\end{bmatrix}\right\} \tag{30}$$

The quantity 'δ' (see (28)) is non-negative in view of (2), (3)-(6) [3, 8]. Observe that, if $\Xi(\alpha_\mu,\beta_\nu) < \mathbf{0}$, then $\Delta V(\boldsymbol{x}(\mu,\nu)) < 0$ for $\boldsymbol{\xi}(\mu,\nu) \neq \mathbf{0}$. Furthermore, $\Delta V(\boldsymbol{x}(\mu,\nu)) = 0$ only when $\boldsymbol{\xi}(\mu,\nu) = 0$. Now, employing Definition 2.1 and following [8], it may be simply demonstrated that $\boldsymbol{x}(\mu,\nu) \longrightarrow \mathbf{0}$ as $\mu \longrightarrow \infty$ and/or $\nu \longrightarrow \infty$ for any boundary constraints fulfilling Equation (3) if $\Delta V(\boldsymbol{x}(\mu,\nu)) < 0$. Thus, the condition $\Xi(\alpha_\mu,\beta_\nu) < \mathbf{0}$ along with (7) is asymptotically stable sufficient condition for the system described by (1)-(6). According to the property of affine matrix functions the condition $\Xi(\alpha_\mu,\beta_\nu) < \mathbf{0}$ if and only if the conditions given in (8) hold true. This concludes the proof of Theorem 3.1. □

4. **Numerical Example.** In this section a numerical example shows the significance of the presented results.

Example 4.1. *Consider the 2-D discrete system described by* (1)-(3) *with*

$$\boldsymbol{A}_1 = \begin{bmatrix}1.2 & -5.5\\ 0.1 & 0\end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix}0.01 & 0\\ 0 & 0.01\end{bmatrix}, \quad \boldsymbol{A}_{d_1} = \boldsymbol{A}_{d_2} = \begin{bmatrix}0 & 0.01\\ 0.01 & 0\end{bmatrix} \tag{31}$$

which includes saturation nonlinearities. Clearly for this example $p_1 = 6.73 > 1$ *and* $p_2 = 0.13 < 1$. *To determine the feasibility of Theorem 3.1, select the matrix* $\boldsymbol{C}$ *of the* *form*

$$\boldsymbol{C} = \begin{bmatrix} g_{12} + h_{12} & \dfrac{g_{12} - h_{12}}{s_2} \\ c_{21} & c_{22} \end{bmatrix} \tag{32}$$

where $g_{12} > 0$*,* $h_{12} > 0$*. Now, our objective is to determine whether the system under consideration satisfies global asymptotic stability condition or not. On solving the LMIs employing MATLAB environment along with YALMIP 3.0 parser* [13] *and SeDuMi 1.21 solver* [14]*, it is found that Theorem 3.1 provides feasible results over the delay ranges* $2 \leq \alpha_\mu \leq 9$ *and* $2 \leq \beta_\nu \leq 7$*.*

It is worth comparing Theorem 3.1 with [1, 4]*. Corollary 1 in* [4] *was able to determine the stability of the above system for the delay range* $2 \leq \alpha_\mu \leq 5$ *and* $2 \leq \beta_\nu \leq 7$*. On comparison it can be observed that the proposed Theorem 3.1 is able to determine the stability over a larger upper delay bound, i.e.,* $\alpha_h = 9$ *while for* [4] $\alpha_h = 5$*. It was also observed that Theorem 1 of* [1] *fails to determine the asymptotic stability of the present system. Therefore, Theorem 3.1 provides better stability results as compared to Corollary 1 of* [4] *and Theorem 1 of* [1] *for the present 2-D system under consideration. The proposed Theorem 3.1, therefore, provides better results as compared to* [1, 4]*.*

5. **Conclusion.** This paper establishes LMI based stability criterion by employing Wirtinger-based integral inequality and reciprocal convexity method. Theorem 3.1 deals with a category of 2-D discrete systems modeled by the FMSLSS model comprising variable delays and saturation nonlinear effects. The effectiveness of the presented results has been demonstrated by an example. The proposed stability results in this paper can be extended for the study of uncertain systems, sensor networks and fuzzy systems.

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