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ANALYSIS OF RELATIONSHIP BETWEEN FINITE MEMORY STRUCTURE SMOOTHER AND FILTER

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Abstract. *In this paper, the relationship between finite memory structure (FMS) smoother and filter is analyzed. Firstly, the FMS smoother is developed by directly solving an optimization problem with the unbiasedness constraint using only finite measurements and inputs on the most recent window. Then, the FMS smoother is shown to be equivalent to existing FMS filters according to the delay length between the measurement and the availability of its estimate. Computer simulation results validate the effectiveness of the FMS smoother and show the relationship between FMS smoother and filter.*
Keywords: Finite memory structure smoother, Finite memory structure filter, Finite impulse response, Filter, Kalman filter, Delay length

1. **Introduction.** As an alternative to the Kalman filter with the infinite memory structure [1-4], the finite memory structure (FMS) filter has been developed for the state estimation and applied successfully for various areas [5-12].

Meanwhile, because the FMS filter is a causal filter providing estimates for states at given times based only on the relative past, the estimates exhibit a delay. Hence, the FMS smoother has been developed for estimation problems where there is a fixed delay between a measurement and the availability of its estimate. FMS smoothers have been derived by solving diverse optimization problems such as best linear unbiased estimation [13,14], robust H_∞ estimation [15], robust H_2 estimation [16], maximum likelihood estimation [17], information filtering estimation [18], weighted least square estimation [19], and Bayesian estimation [20]. Although these FMS smoothers have their own unique features, they have the following common advantages. The smoother generally utilizes more measurement information than the filter to provide state estimates, which can give more accurate estimation performance than the filter. In addition, since the smoother provides state estimates at the delayed time using measurement information up to the current time, measurement information can be reflected in advance in the presence of the state change, which can give faster convergence than the filter.

In this paper, the relationship between finite memory structure smoother and filter is analyzed. Firstly, the FMS smoother is developed by directly solving an optimization problem with the unbiasedness constraint using only finite measurements and inputs on the most recent window. Then, the FMS smoother is shown to be equivalent to the FMS filter [11,12], the batch unbiased finite impulse responses (FIR) filter [6], and the backward FMS filter [5] according to the delay length between the measurement and the availability of its estimate. Computer simulation results for a noisy sinusoidal signal model validate the effectiveness of the FMS smoother and show the relationship between FMS smoother and filter.

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This paper has the following structure. In Section 2, a discrete-time FMS smoother is developed. In Section 3, the FMS smoother is shown to be equivalent to existing FMS filters. In Section 4, computer simulations are performed. Then, concluding remarks are given in Section 5.

2. **Discrete-Time FMS Smoother.** The discrete-time state-space model with control input u_i can be represented by

$$\begin{aligned} x_{i+1} &= Ax_i + Bu + Gw_i, \\ z_i &= Cx_i + v_i, \end{aligned} \tag{1}$$

where x_i is the state, and z_i is the measurement. The initial state $\hat{x}_{i_0}$ is a random variable with a mean $\bar{x}_{i_0}$ and a covariance Σ_{i_0}. The system noise w_i and the measurement noise v_i are zero-mean white Gaussian whose covariances Q and R are assumed to be positive definite matrix.

FMS smoothers have been derived by solving diverse optimization problems [13-20]. Among them, in this section, the FMS smoother to estimate the state x_{i-d} at the lagged time $i-d$ is developed by directly solving an optimization problem with the unbiasedness constraint using only finite measurements and inputs on the most recent window $[i-M, i]$. The lagged time $i-d$ means there is a fixed delay between the measurement and the availability of its estimate. The positive integer d is the delay length satisfying $0 \le d < M$ and equal to the number of discrete time steps between the lagged time $i-d$ at which the state is to be estimated and the current time i of the last measurement used in estimating it. The window initial time $i-M$ will be denoted by i_M hereafter for simplicity. Finite measurements and inputs on the most recent window $[i_M, i]$ are denoted by Z_i and U_i, respectively, and represented by

$$\begin{aligned} Z_i &\triangleq \left[z_{i_M}^T \; z_{i_M+1}^T \cdots z_{i-1}^T\right]^T, \\ U_i &\triangleq \left[u_{i_M}^T \; u_{i_M+1}^T \cdots u_{i-1}^T\right]^T. \end{aligned}$$

Using Z_i and U_i, the discrete-time state-space model (1) can be represented in the following regression form

$$Z_i - \bar{\Xi} U_i = \bar{\Gamma} x_{i_M} + \bar{\Lambda} W_i + V_i, \tag{2}$$

where W_i and V_i have the same form as (2) for w_i, v_i, respectively, and matrices $\bar{\Gamma}$, $\bar{\Xi}$, $\bar{\Lambda}$ are as follows:

$$\bar{\Gamma} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{M-2} \\ CA^{M-1} \end{bmatrix},$$

$$\bar{\Xi} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CB & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{M-3}B & CA^{M-4}B & \cdots & 0 & 0 \\ CA^{M-2}B & CA^{M-3}B & \cdots & CB & 0 \end{bmatrix},$$

$$\bar{\Lambda} \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CG & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{M-3}G & CA^{M-4}G & \cdots & 0 & 0 \\ CA^{M-2}G & CA^{M-3}G & \cdots & CG & 0 \end{bmatrix}.$$

From the discrete-time state-space model (1), the state x_{i-d} at the lagged time $i-d$ is represented by

$$x_{i-d} = A^{M-d}x_{i_M} + \tilde{\Xi}U_i + \tilde{\Lambda}W_i, \tag{3}$$

where

$$\tilde{\Xi} \triangleq \left[A^{M-d-1}B \ \cdots \ AB \ B \ \overbrace{0\ 0\ \cdots\ 0}^{d} \right],$$

$$\tilde{\Lambda} \triangleq \left[A^{M-d-1}G \ \cdots \ AG \ G \ \overbrace{0\ 0\ \cdots\ 0}^{d} \right]. \tag{4}$$

Therefore, using (3), the regression form (2) can be expressed in terms with x_{i-d} at the lagged time $i-d$ as follows:

$$Z_i - \Xi U_i = \Gamma x_{i-d} + \Lambda W_i + V_i, \tag{5}$$

where

$$\Gamma \triangleq \bar{\Gamma}A^{-(M-d)}, \quad \Lambda \triangleq \bar{\Lambda} - \bar{\Gamma}A^{-(M-d)}\tilde{\Lambda}, \quad \Xi \triangleq \bar{\Xi} - \bar{\Gamma}A^{-(M-d)}\tilde{\Xi}. \tag{6}$$

The noise term $\Lambda W_i + V_i$ in (5) is zero-mean white Gaussian with covariance Π given by

$$\Pi \triangleq \Lambda \left[\mathrm{diag}(\overbrace{Q\ Q\ \cdots\ Q\ Q}^{M}) \right] \Lambda^T + \left[\mathrm{diag}(\overbrace{R\ R\ \cdots\ R\ R}^{M}) \right],$$

where $\mathrm{diag}(Q\ Q\ \cdots\ Q\ Q)$ and $\mathrm{diag}(R\ R\ \cdots\ R\ R)$ denote block-diagonal matrices with M elements of Q and R, respectively.

The FMS smoother is developed from *best linear unbiased estimation* approach in [21]. The FMS smoother $\hat{x}_{i-d}$ is assumed to be obtained from only finite measurements Z_i and inputs U_i on the most recent window $[i_M, i]$ as follows:

$$\hat{x}_{i-d} \triangleq \mathcal{H}(Z_i - \Xi U_i), \tag{7}$$

where $\mathcal{H}$ is the gain matrix. Taking the expectation of both sides of (7), the following relation is obtained:

$$\mathcal{E}[\hat{x}_{i-d}] = \mathcal{E}[\mathcal{H}(Z_i - \Xi U_i)] = \mathcal{H}\Gamma\mathcal{E}[x_{i-d}],$$

where $\mathcal{E}[x]$ denotes the expectation of x. Then, with the following constraint:

$$\mathcal{H}\Gamma = I, \tag{8}$$

$\hat{x}_{i-d}$ is unbiased, i.e., $\mathcal{E}[\hat{x}_{i-d}] = \mathcal{E}[x_{i-d}]$. Thus, the constraint (8) can be called the *unbiasedness constraint* for the FMS smoother $\hat{x}_{i-d}$.

The objective is now to obtain the gain matrix $\mathcal{H}_*$, subject to the unbiasedness constraint (8), in such a way that the error of $\hat{x}_{i-d}$ has a minimum variance as follows:

$$\mathcal{H}_* = \arg\min_{\mathcal{H}} \mathcal{E}\left[(x_{i-d} - \hat{x}_{i-d})^T (x_{i-d} - \hat{x}_{i-d}) \right]. \tag{9}$$

Using the approach of best linear unbiased estimation in [21], the FMS smoother $\hat{x}_{i-d}$ is obtained by the solution of (9) as follows:

$$\hat{x}_{i-d} = \mathcal{H}(Z_i - \Xi U_i), \tag{10}$$

where

$$\mathcal{H} = \left(\Gamma^T \Pi^{-1} \Gamma \right)^{-1} \Gamma^T \Pi^{-1}.$$

The FMS smoother $\hat{x}_{i-d}$ (10) has several inherent properties such as unbiasedness, deadbeat, time-invariance and robustness as follows. The unbiasedness of the state estimate means that its mean value tracks the mean value of the state at every time for noisy systems. The deadbeat of the estimate means that its value tracks exactly the state at every time for noise-free systems. This deadbeat property indicates finite convergence time and fast tracking ability of the FMS smoother. Thus, it can be expected that the FMS smoother would be appropriate for fast estimation and detection of signals with unknown times of occurrence. In addition, as shown in (10), the gain matrix $\mathcal{H}$ for the FMS smoother requires computation only on the interval $[0, M]$ once and is time-invariant for all windows. The on-line computation of the FMS smoother requires only filter updates. Hence, the computational complexity of the FMS smoother is $\mathcal{O}(M)$ and thus linear in the size of the window length M. In practice, this means that quite a large M can be chosen without worrying about computational burden. In addition, unlike the fixed-lag Kalman smoother, the FMS smoother does not need the combining of two algorithms for before and after estimated time $i - d$, the memory requirement for the save of intermediate values, and the initialization algorithm. Moreover, the FMS smoother might be robust against temporary uncertainties such as model uncertainty, unknown input, and incomplete measurement, due to its finite memory structure. This intrinsic robustness will be verified later through extensive computer simulations.

3. **Relationship between Finite Memory Structure Smoother and Filter.**

In this section, according to the delay length d between the measurement and the availability of its estimate, the FMS smoother is shown to be equivalent to existing FMS filters [5,6,11,12].

3.1. **When there is no delay.**

When there is no delay, that is $d = 0$, matrices of (4) become

$$\tilde{\Xi} = \left[A^{M-1}B \; A^{M-2}B \; \cdots \; AB \; B\right], \\ \tilde{\Lambda} = \left[A^{M-1}G \; A^{M-2}G \; \cdots \; AG \; G\right], \tag{11}$$

and thus matrices of (6) become

$$\Gamma = \bar{\Gamma}A^{-M} = \begin{bmatrix} CA^{-M} \\ CA^{-M+1} \\ \vdots \\ CA^{-2} \\ CA^{-1} \end{bmatrix} \triangleq \check{\Gamma},$$

$$\Lambda = \bar{\Lambda} - \bar{\Gamma}A^{-M}\tilde{\Lambda} = -\begin{bmatrix} CA^{-1}G & CA^{-2}G & \cdots & CA^{-M}G \\ 0 & CA^{-1}G & \cdots & CA^{-M+1}G \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & CA^{-2}G \\ 0 & 0 & \cdots & CA^{-1}G \end{bmatrix} \triangleq \check{\Lambda},$$

$$\Xi = \bar{\Xi} - \bar{\Gamma}A^{-M}\tilde{\Xi} = -\begin{bmatrix} CA^{-1}B & CA^{-2}B & \cdots & CA^{-M}B \\ 0 & CA^{-1}B & \cdots & CA^{-M+1}B \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & CA^{-2}B \\ 0 & 0 & \cdots & CA^{-1}B \end{bmatrix} \triangleq \check{\Xi}. \tag{12}$$

Then, the FMS smoother (10) with $d = 0$ can be represented by

$$\hat{x}_i = \left(\check{\Gamma}^T\check{\Pi}^{-1}\check{\Gamma}\right)^{-1}\check{\Gamma}^T\Pi^{-1}\left(Z_i - \check{\Xi}U_i\right) \tag{13}$$

with matrices (12) and

$$\check{\Pi} \triangleq \check{\Lambda}\left[\mathrm{diag}(\overbrace{Q \; Q \; \cdots \; Q \; Q}^{M})\right]\check{\Lambda}^T + \left[\mathrm{diag}(\overbrace{R \; R \; \cdots \; R \; R}^{M})\right].$$

Therefore, the FMS smoother (13) with $d = 0$ is equivalent to the existing FMS filter [11,12].

3.2. **When there is no delay and the covariance of the noise term has an identity matrix.** When there is no delay, that is $d = 0$, and the covariance Π of the noise term $\Lambda W_i + V_i$ in (5) has an identity matrix, that is $\Pi = I$, the FMS smoother (10) can be represented by

$$\hat{x}_i = \left(\check{\Gamma}^T\check{\Gamma}\right)^{-1}\check{\Gamma}^T\left(Z_i - \check{\Xi}U_i\right). \tag{14}$$

Thus, the FMS smoother with $d = 0$ and $\Pi = I$ is shown to be equivalent to the existing batch unbiased FIR filter [6].

3.3. **When the delay length is the same with the window length.** When the delay length is the same with the window length, that is $d = M$, matrices of (4) become

$$\tilde{\Xi} = \left[\overbrace{0\ 0\ \cdots\ 0\ 0}^{d}\right], \quad \tilde{\Lambda} = \left[\overbrace{0\ 0\ \cdots\ 0\ 0}^{d}\right], \tag{15}$$

and thus matrices of (6) become

$$\Gamma = \bar{\Gamma}, \quad \Lambda = \bar{\Lambda}, \quad \Xi = \bar{\Xi}. \tag{16}$$

Then, the FMS smoother with $d = M$ can be represented by

$$\hat{x}_{i-M} = \left(\bar{\Gamma}^T\bar{\Pi}^{-1}\bar{\Gamma}\right)^{-1}\bar{\Gamma}^T\bar{\Pi}^{-1}\left(Z_i - \bar{\Xi}U_i\right), \tag{17}$$

with matrices (16) and

$$\bar{\Pi} \triangleq \bar{\Lambda}\left[\mathrm{diag}(\overbrace{Q\ Q\ \cdots\ Q\ Q}^{M})\right]\bar{\Lambda}^T + \left[\mathrm{diag}(\overbrace{R\ R\ \cdots\ R\ R}^{M})\right].$$

Therefore, the FMS smoother (17) with $d = M$ is equivalent to the existing backward FMS filter [5].

4. **Computer Simulations for Sinusoidal Signal Model.** In this section, the FMS smoother is applied for the noisy sinusoidal signal model through computer simulations. In order to verify intrinsic robustness property of the FMS smoother, the noisy sinusoidal signal model is assumed to have a temporary model uncertainty as follows:

$$A = \begin{bmatrix} \cos(\pi/32) + \delta_i & \sin(\pi/32) \\ -\sin(\pi/32) & \cos(\pi/32) + \delta_i \end{bmatrix},$$
$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} \cos(\pi/4) + 0.2\delta_i & 0.2\delta_i \end{bmatrix}, \tag{18}$$

where the uncertain model parameter δ_i is taken by

$$\delta_i = \begin{cases} 0.08 & \text{if } 100 \le i \le 150, \\ 0 & \text{otherwise.} \end{cases}$$

The system noise covariance and the measurement noise covariance are taken by $Q = 0.05^2$ and $R = 0.05^2$, respectively. The FMS smoother with $M = 15$ and $d = 3$, the FMS filter with $M = 15$ and the Kalman filter are compared. Although three filters are computed by the discrete-time state-space model (1) without the consideration of the temporary model uncertainty, actual measurements and inputs for these estimation filters are obtained from the actual system with the temporary model uncertainty (18). To make a clearer comparison of estimation performances, simulations of 20 runs are performed using different system and observation noises, and each single simulation run lasts 500 samples.

Figure 1 shows root mean square (RMS) estimation errors of the 1st state for 20 simulations and also shows estimation errors for one of 20 simulations. Even if the sinusoidal signal model is accurately represented in state-space model on a long time scale, unpredictable changes such as frequency, phase, and speed jumps can occur. This effect is called temporary uncertainties because it generally occurs in the short term. There can be a model uncertainty, an unknown input, and incomplete measurement information, etc., as representative temporary uncertainties. In simulations, the model uncertainty is considered. As shown in simulation results, the FMS smoother can be better than other two filters in terms of error magnitude and error convergence. The magnitude of estimation error of FMS smoother is smaller than that of other two filters on the interval where the temporary model uncertainty exists. In addition, the convergence of estimation error of FMS smoother is faster than that of other two filters after the temporary model uncertainty disappears. Therefore, the FMS smoother can be more robust than other two filters when applied to the noisy sinusoidal signal model with the temporary model uncertainty, although the FMS smoother is designed with no consideration of robustness. This observation means that the FMS smoother has an intrinsic robustness property. Meanwhile, the FMS smoother can be comparable to other two filters after the effect of the temporary model uncertainty completely disappears. Figure 2 shows that the equivalent relationship between FMS smoother and filter when there is no delay.

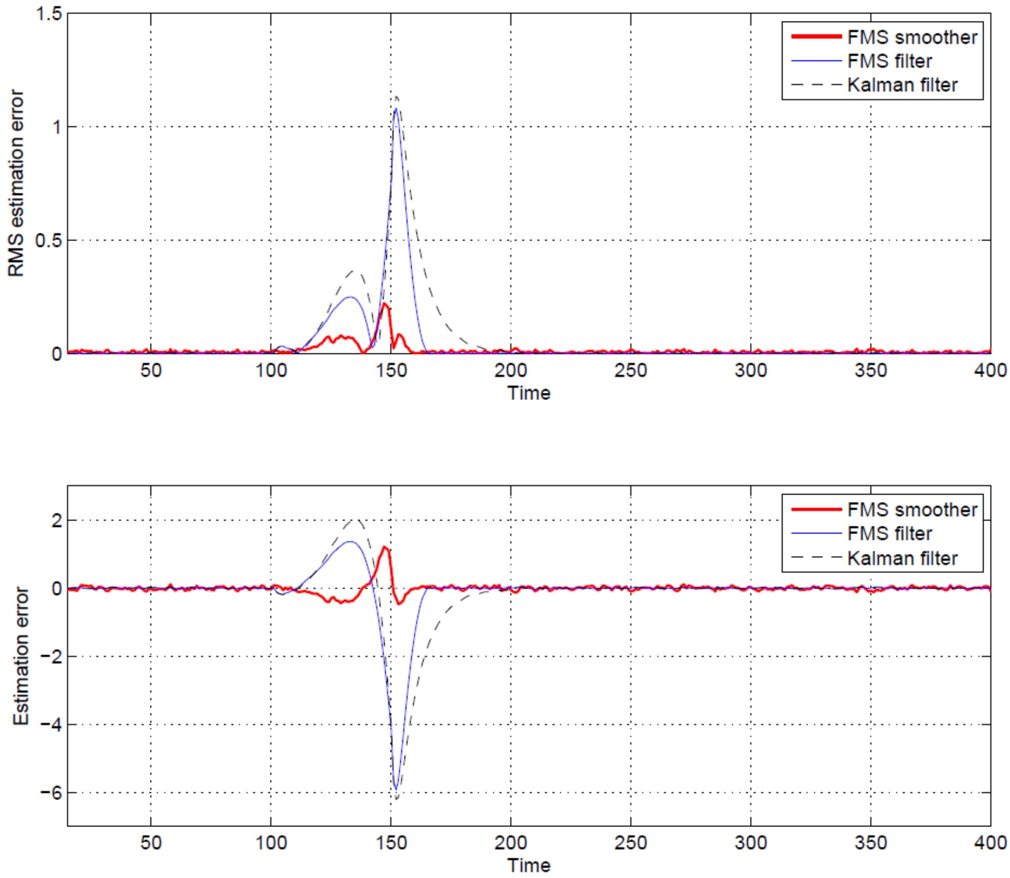

FIGURE 1. Estimation errors

5. **Concluding Remarks.** This paper has analyzed the relationship between finite memory structure smoother and filter. The FMS smoother has been developed by directly solving an optimization problem with the unbiasedness constraint using only finite measurements and inputs on the most recent window. The FMS smoother has been then shown to be equivalent to the FMS filter, the batch unbiased FIR filter, and the backward

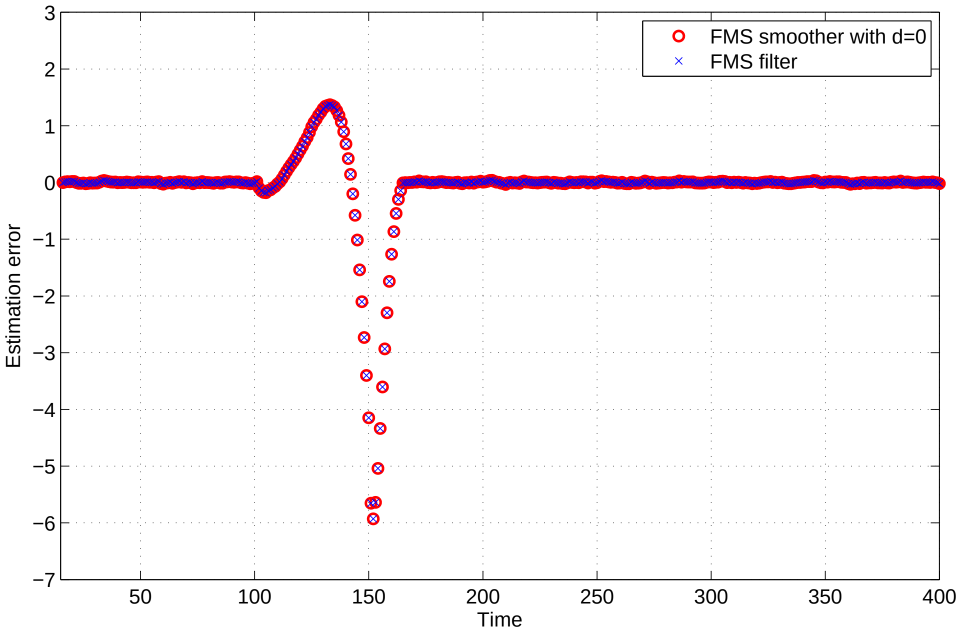

Figure 2. Equivalent relationship between FMS smoother and filter

FMS filter according to the delay length between the measurement and the availability of its estimate. Through computer simulations for a noisy sinusoidal signal model, the effectiveness of the FMS smoother has been validated and the relationship between FMS smoother and filter has been verified. An alternative FMS smoother for nonzero-mean Gaussian noises can be researched as future work because the research work on this case is relatively inactive.

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