

SEMI-PRIME UP-FILTERS IN MEET-COMMUTATIVE UP-ALGEBRAS

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ABSTRACT. *The concept of semi-prime UP-filters in meet-commutative UP-algebras, different from the concept of prime UP-filters in such algebras, was introduced and analyzed. While in a meet-commutative UP-algebra, the intersection of prime UP-filters does not have to be a prime UP-filters, the intersection of semi-prime UP-filters in such UP-algebra is a semi-prime UP-filter always. Additionally, another specificity of meet-commutative UP-algebras is illustrated.*

Keywords: UP-algebra, Meet-commutative UP-algebra, Prime UP-filter, Semi-prime UP-filter

1. Introduction. There is a belief in the academic community of mathematicians that an important task of artificial intelligence is to make a computer simulate a human being in dealing with certainty and understanding of uncertainty in information. They are convinced that in the human endeavor to solve this task logic can offer a foundation for these intentions. Until recently, information processing dealing with certain information relied on classical logic. Non-classical logic including many-valued logic such as intuitionist logic and fuzzy logic takes the advantage of classical logic to handle information with various faces of uncertainty, such as fuzziness and randomness. Therefore, non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Such logics have been represented as algebras, that is, sets with one, two, or more algebraic operations satisfying some conditions inspired by these logics. In the past and this century, many mathematicians have tried to design algebraic systems such as BCK-algebras, BCI-algebras, BL-algebras, KU-algebras, and UP-algebras.

Filters theory plays an important role in studying any class of logical algebras. From the logical point of view, various filters correspond to various sets of valid formulas in appropriate logic. On the other hand, designing different types of filters in some logical algebra is also interesting from an algebraic aspect.

The concept of KU-algebras was introduced in 2009 by Prabpayak and Leerawat in [1]. In 2017, Iampan [2] introduced the concept of UP-algebras as a generalization of KU-algebras. In [3], Somjanta et al. introduced the notion of filters in this class of algebras. Jun and Iampan then introduced and analyzed several classes of filters in UP-algebras such as near UP-filter, implicative, comparative and shift UP-filters (see, for example, [4, 5, 6, 7]). Romano also took part in the analysis of filter types in such algebras as proper UP-filters [8] and (with Jun) weak implicative UP-filters [9].

The concept of meet-commutative UP-algebras was introduced in [10] by Sawika et al. In such algebras, Muhiuddin et al. introduced the concepts of prime (of the first kind) and irreducible UP-filters [11]. In that paper it is also shown that any prime UP-filter in a meet-commutative UP-algebra is an irreducible UP-filter. Also, in this article it is shown that the intersection of prime UP-filters does not have to be a prime UP-filter. Continuing to develop the ideas introduced in the mentioned paper [11], Romano introduced the concepts of prime UP-filters of the second [12] and the third kind [13]. This seems to justify the author's interest in studying the properties of UP-algebras in which the property of meet-commutativity is present.

Since (A, \sqcup) , where A is a meet-commutative UP-algebra and " $x \sqcup y$ " is the least upper bound of x and y , is a semi-lattice, researchers of this class of logical algebras seek to assess whether it is possible and how to design a modification of Stone's theorem from 1936 [24]. To this end, possible candidates will be explored who will play a crucial role in designing the variation of this theorem for the class of meet-commutative UP-algebras.

In this paper, as a continuation of the previous research, the authors introduce and analyze the new concept of semi-prime UP-filters in meet-commutative UP-algebras which is different from the concept of prime UP-filters. Some examples and necessary conditions for both concepts are provided. While in a meet-commutative UP-algebra, the intersection of prime UP-filters does not have to be a prime UP-filters, the intersection of semi-prime UP-filters in such UP-algebra is a semi-prime UP-filter always. Additionally, another specificity of meet-commutative UP-algebras is illustrated.

2. Preliminaries. This section introduces the concepts and processes with those that will be used in the main section.

2.1. UP-algebras. In this subsection, taking from the literature, we will repeat the logical environment of interest for this research.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* (see [2]) if it satisfies the following axioms:

- (UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
- (UP-2) $(\forall x \in A)(0 \cdot x = x)$,
- (UP-3) $(\forall x \in A)(x \cdot 0 = 0)$,
- (UP-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

Example 2.1 ([2], Example 1.6). *Let $A = \{0, a, b, c\}$ and a binary operation " \cdot " is defined on A as follows:*

\cdot	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $A = (A, \cdot, 0)$ is a UP-algebra.

In a UP-algebra $A = (A, \cdot, 0)$, the order relation " \leq " is defined on A as follows:

$$(\forall x, y \in A)(x \leq y \iff x \cdot y = 0).$$

Some of the important features of this inequality are given by the following proposition.

Proposition 2.1 ([2]). *Let A be a UP-algebra. Then*

- a) $(\forall x \in A)(x \leq 0)$,
- b) $(\forall x, y \in A)(x \leq y \cdot x)$,
- c) $(\forall x, y, z \in A)(y \cdot z \leq (x \cdot y) \cdot (x \cdot z))$,
- d) $(\forall x, y, z \in A)(y \cdot z \leq (z \cdot x) \cdot (y \cdot x))$.

2.2. UP-filters. A subset F of a UP-algebra A is called a *UP-filter* of A (see [3]) if it satisfies the following conditions:

- (F-1) $0 \in F$,
- (F-2) $(\forall x, y \in A)((x \in F \wedge x \cdot y \in F) \implies y \in F)$.

It is clear that every UP-filter F of a UP-algebra A satisfies:

- (1) $(\forall x, y \in A)((x \in F \wedge x \leq y) \implies y \in F)$.

The family of all UP-filters of a UP-algebra A is denoted by $\mathfrak{F}(A)$. It is easy to verify that the intersection of UP-filters of a UP-algebra A is also a UP-filter of A . For any subset S of A , let $F(S) := \bigcap \{F \in \mathfrak{F}(A) : S \subseteq F\}$. Then $F(S)$ is the smallest UP-filter of A containing S . Therefore, if S and T are UP-filters in a UP-algebra A , then $S \sqcap T := S \cap T$ and $S \sqcup T := F(S \cup T)$ are UP-filters of A . So, $(\mathfrak{F}(A), \sqcap, \sqcup)$ is a complete lattice.

2.3. Meet-commutative UP-algebras. A UP-algebra A is said to be *meet-commutative* (see [10], Definition 1.15) if it satisfies the condition

- (2) $(\forall x, y \in A)((x \cdot y) \cdot y = (y \cdot x) \cdot x)$.

However, this term also appears (Definition 3) in [6]. We begin this subsection with the following example.

Example 2.2. Let $A = \{0, a, b\}$ and a binary operation “.” is defined on A as follows:

.	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra ([10], Example 1.16).

The following example shows that any UP-algebra may not satisfy the condition (2).

Example 2.3 ([6], Example 3). Let $A = \{0, a, b, c\}$ and a binary operation “.” is defined on A as follows:

.	0	a	b	c
0	0	a	b	c
a	0	0	a	a
b	0	0	0	a
c	0	0	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra which does not satisfy the condition (2). For example, for $x = b$ and $y = c$, we have $(x \cdot y) \cdot y = (b \cdot c) \cdot c = a \cdot c = a$ but $(y \cdot x) \cdot x = (c \cdot b) \cdot b = 0 \cdot b = b$.

We first characterize the meet-commutative UP-algebras.

Theorem 2.1 ([11]). Let A be a meet-commutative UP-algebra. Then the following holds:

$$(\forall x, y \in A)(x \leq y \implies y = (y \cdot x) \cdot x).$$

The following theorem is crucial for understanding this logic environment and recognizing the properties of various types of UP-filters in such an environment.

Theorem 2.2 ([11]). Let A be a meet-commutative UP-algebra. For any $x, y \in A$, the element

$$(3) \ x \sqcup y := (x \cdot y) \cdot y = (y \cdot x) \cdot x$$

is the least upper bound of x and y .

Theorem 2.3 ([12, 13]). Let A be a meet-commutative UP-algebra. Then

- a) $(\forall x, y \in A)(0 \sqcup x = x, \ x \sqcup 0 = 0, \ x \sqcup x = x, \ \text{and} \ x \sqcup y = y \sqcup x)$,
- b) $(\forall x, y, z \in A)((x \sqcup y) \sqcup z = (x \sqcup z) \sqcup (y \sqcup z))$,
- c) $(\forall x, y, z \in A)((z \cdot x) \sqcup (z \cdot y) \leq z \cdot (x \sqcup y))$,

- d) $(\forall x, y, z \in A)((x \sqcup y) \cdot z \leq (x \cdot z) \sqcup (y \cdot z)),$
e) $(\forall x, y \in A)(x \sqcup y \leq (y \cdot x) \sqcup (x \cdot y)).$

Corollary 2.1. *If A is a meet-commutative UP-algebra, then (A, \sqcup) is the upper semi-lattice.*

2.4. Prime UP-filters. The concept of prime UP-filters in meet-commutative UP-algebras was introduced and analyzed in [11] by Muhiuddin et al.

Definition 2.1 ([11]). *Let F be a UP-filter of A . Then F is said to be a prime UP-filter of A if the following holds:*

$$(PF) (\forall x, y \in A)(x \sqcup y \in F \implies (x \in F \vee y \in F)).$$

Example 2.4. *Let $A = \{0, a, b, c\}$ and a binary operation “ \cdot ” is defined on A as follows:*

\cdot	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	c	0	c
c	0	b	b	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra. Subsets $\{0\}$, $\{0, b\}$, $\{0, c\}$ and $\{0, a, b, c\}$ are UP-filters of A . It is not difficult to verify that UP-filters $\{0, b\}$ and $\{0, c\}$ are prime. It is clear that $\{0\}$ is not a prime UP-filter of A because $b \sqcup c = 0 \in \{0\}$ but $b \notin \{0\}$ and $c \notin \{0\}$.

Note 2.1. *Let us note that this previous example shows that in a meet-commutative UP-algebra A the intersection of prime UP-filters of A does not have to be a prime UP-filter of A . For prime UP-filters $\{0, b\}$ and $\{0, c\}$ it is $\{0, b\} \cap \{0, c\} = \{0\}$ but $\{0\}$ is not a prime UP-filter.*

3. The Concept of Semi-Prime UP-Filters. The term of semi-prime ideals was first used by Krull in his famous paper [14] (p.735). (Cite according to [15], p.107.). Rav adapted this term to general lattices ([15]). In addition, in the mentioned article, Rav introduced and analyzed the concept of semi-prime filters in general lattices, too. These concepts have been studied in several research papers (see, for example, [16, 17, 18, 19, 20, 21]).

Definition 3.1. *Let A be a meet-commutative UP-algebra. A UP-filter F of A is said to be a semi-prime UP-filter of A if the following holds:*

$$(\forall x, y, z \in A)((x \sqcup y \in F \wedge x \sqcup z \in F) \implies (\exists d \in A)(d \leq y \wedge d \leq z)(x \sqcup d \in F)).$$

The following example shows that one UP-filter in a meet-commutative UP-algebra can be a prime UP-filter and a semi-prime UP-filter at the same time.

Example 3.1. *Let A be a meet-commutative UP-algebra as in Example 2.4. By direct checking, it can be proved that the subsets $F := \{0, b\}$ and $G := \{0, c\}$ are semi-prime UP-filters of A and they are prime UP-filters (see Example 2.4).*

A UP-filter in a meet-commutative UP-algebra can be a semi-prime UP-filter and it is not a prime UP-filter, as the following example shows.

Example 3.2. *Let A be a meet-commutative UP-algebra as in Example 2.4. By direct checking, it can be proved that the subset $\{0\}$ is a semi-prime UP-filter of A but it is not a prime UP-filter as it is shown in Example 2.4.*

Proposition 3.1. *If a meet-commutative UP-algebra A contains at least one semi-prime UP-filter, then the algebra A has the following property*

$$(DfB) (\forall a, b \in A)(\exists c \in A)(c \leq a \wedge c \leq b).$$

Proof: Let $a, b \in A$ and let F be a semi-prime UP-filter of A . Then $x \sqcup a \in F$ and $x \sqcup b \in F$ for all $x \in F$. Since F is a semi-prime UP-filter in A , there exists an element $d \in A$ such that $d \leq a$ and $d \leq b$. Hence, A satisfies the required condition. \square

It follows from the previous proposition that if we want to observe semi-prime UP-filters in a meet-commutative UP-algebra, we must assume that this UP-algebra satisfies the condition (DfB). This comment is a justification for the following definition.

Definition 3.2. *A meet-commutative UP-algebra is called directed from below meet-commutative UP-algebra if it satisfies the condition (DfB).*

While in a meet-commutative UP-algebra, the intersection of prime UP-filters does not have to be a prime UP-filters (see Note 2.1), the intersection of semi-prime UP-filters in such UP-algebra is a semi-prime UP-filter always.

Theorem 3.1. *The intersection of two semi-prime UP-filters of a meet-commutative UP-algebra A is a semi-prime UP-filter of A .*

Proof: Let F_1 and F_2 be semi-prime UP-filters in a meet-commutative UP-algebra A . It is clear that the intersection $F_1 \cap F_2$ satisfies conditions (F-1) and (F-2). Let $x, y, z \in A$ be arbitrary elements such that $x \sqcup y \in F_1 \cap F_2$ and $x \sqcup z \in F_1 \cap F_2$. Since F_1 and F_2 are semi-prime UP-filters of A , there exists an element $d_1 \in A$ such that $d_1 \leq y \wedge d_1 \leq z \wedge x \sqcup d_1 \in F_1$ and there exists an element $d_2 \in A$ such that $d_2 \leq y \wedge d_2 \leq z \wedge x \sqcup d_2 \in F_1$. Then $x \sqcup (d_1 \sqcup d_2) \in F_1$ and $x \sqcup (d_1 \sqcup d_2) \in F_2$ by (1). So, $x \sqcup (d_1 \sqcup d_2) \in F_1 \cap F_2$. As $d_1 \sqcup d_2 \leq y$ and $d_1 \sqcup d_2 \leq z$ hold, we conclude that $F_1 \cap F_2$ is a semi-prime UP-filter of A . \square

Example 3.3. *Let A be as in Example 2.4. Subsets $F_1 := \{0\}$, $F_2 := \{0, b\}$, $F_3 := \{0, c\}$ and $F_4 := A$ are semi-prime UP-filters of A . By direct checking, it can be determined that each intersection of some (or all) of these UP-filters is a semi-prime UP-filter as well. For example, $\cap_i F_i = F_1$, $F_2 \cap F_3 = F_1$ and $F_2 \cap F_4 = F_2$.*

Note 3.1. *It can be proved that the intersection of any final family of semi-prime UP-filters in a meet-commutative UP-algebra A is a semi-prime UP-filter of A , too.*

To the question “When will a prime UP-filter F in a meet-commutative UP-algebra A be a semi-prime UP-filter?”, one of the possible answers is given by the following theorem.

Theorem 3.2. *Let A be a meet-commutative UP-algebra A which satisfies the condition (DfB). If a prime UP-filter F of A satisfies the condition*

$$(F-3) (\forall a, b \in F)(\exists d \in F)(d \leq a \wedge d \leq b),$$

then it is a semi-prime UP-filter of A .

Proof: Let $a, b, x \in A$ be elements such that $a \sqcup b \in F$ and $a \sqcup c \in f$. Then $a \in F \vee b \in F$ and $a \in F \vee c \in F$ by (PF). We have two options.

i) $a \in F$. On the other hand, for the elements b and c , there exists an element $d \in A$ such that $d \leq b$ and $d \leq c$ by (DfB). As $a \sqcup d \in F$ holds, it is clear that the condition of Definition 3.1 is satisfied.

ii) $a \notin F$ and $b \in F \wedge c \in F$. Then there exists an element $d \in F$ such that $d \leq b \wedge d \leq c$ according to (F-3). Thus, $a \sqcup d \in F$ by $d \leq a \sqcup d$ and (1). So, we have shown that the condition of Definition 3.1 is again satisfied.

This shows that F is a semi-prime UP-filter of A . \square

The following example shows that a prime UP-filter in a meet-commutative UP-algebra does not have to be a semi-prime UP-filter.

Example 3.4. Let $A = \{0, a, b, c, d\}$ and a binary operation “ \cdot ” is defined on A as follows:

\cdot	0	a	b	c	d
0	0	a	b	c	d
a	0	0	0	0	d
b	0	c	0	c	d
c	0	b	b	0	d
d	0	a	b	c	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra. Subsets $\{0\}$, $\{0, b\}$, $\{0, c\}$, $\{0, d\}$ and $\{0, a, b, c, d\}$ are UP-filters of A . In doing so, the following are valid:

- The UP-filters $\{0\}$, $\{0, b\}$ and $\{0, c\}$ are not prime and they are not semi-prime,
- The UP-filter $F := \{0, d\}$ is a prime UP-filter of A but it is not a semi-prime since this UP-algebra does not satisfy the condition (DfB). For example, no lower bound can be found for the elements $a, d \in A$.

In what follows, we will illustrate one specificity of the meet-commutative UP-algebras. Let A be a meet-commutative UP-algebra and let F be a non-empty subset of A and $x \in A$ be an arbitrary element. The notation $x^{-1}F$ for the set $\{y \in A : x \sqcup y \in F\}$ is borrowed from [22] (Definition 3.2). In the mentioned paper, it is shown (Proposition 3.3) that if L is a lattice implication algebra and F is a filter in L , then $x^{-1}F$ is a filter in L also. [20] shows (Theorem 8) something more: Let S be a directed from below join semilattice with 1 and F be a filter of S . If F is a semi-prime filter in S , then $x^{-1}F$ is a semi-prime filter in S , too. In the case of meet-commutative UP-algebras, if F is a UP-filter, the subset $x^{-1}F$ does not have to be a UP-filter at all. For example, for the UP-filter $F := \{0\}$ in Example 3.4, we have $c^{-1}F = \{0, b, d\}$ and this last set is not UP-filter at all.

4. Final Comments and Possible Further Work. The concept of prime UP-filters of meet-commutative UP-algebras was introduced and analyzed by Muhiuddin et al. in [11]. The concept of semi-prime UP-filters in such algebras was introduced in this paper. Also, it has been shown that this term differs from prime UP-filters. Additionally, some specifics of this newly introduced concept in meet-commutative UP-algebras are shown.

The semi-lattice (A, \sqcup) , where A is a meet-commutative UP-algebra and $a \sqcup b$ the least upper bound of A for the elements a and b , differs significantly from the standard semi-lattices that appear in the literature. In some of the future research, one could, among other things, try to find some of the specifics of this semi-lattice. Moreover, the authors plan to introduce the concepts of bipolar fuzzy prime UP-filters and bipolar fuzzy semi-prime UP-filters in UP-algebras and study it in the same way as in [23].

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