ICIC Express Letters
Volume **16**, Number **10**, October **2022**

ICIC International ©2022 ISSN 1881-803X
pp. 1057–1062

ESSENTIAL UP-FILTERS AND t-ESSENTIAL FUZZY UP-FILTERS OF UP-ALGEBRAS

Thiti Gaketem[1], Pannawit Khamrot[2] and Aiyared Iampan[1,*]

[1]Fuzzy Algebras and Decision-Making Problems Research Unit
Department of Mathematics
School of Science
University of Phayao
19 Moo 2, Tambon Mae Ka, Amphur Mueang, Phayao 56000, Thailand
thiti.ga@up.ac.th; *Corresponding author: aiyared.ia@up.ac.th

[2]Faculty of Science and Agricultural Technology
Ralamangala University of Technology Lanna Phitsanulok
52 Moo 7, Tambon Ban Krang, Amphur Mueang, Phitsanulok 65000, Thailand
pk_g@rmutl.ac.th

Received November 2021; accepted January 2022

Abstract. *In this article, we introduce the new concept of essential UP-filters from the concept of essential UP-ideals in UP-algebras. Although we know that the concept of UP-ideals is UP-filters, we have an example showing that the concept of essential UP-ideals is not necessarily essential UP-filters. After that, we extend the concept of essential UP-filters to t-essential fuzzy UP-filters of UP-algebras and study their relationship based on characteristic functions and upper level subsets.*
Keywords: UP-algebra, Essential UP-ideal, Essential UP-filter, t-essential fuzzy UP-filter

1. **Introduction.** Currently, researchers have proposed several types of algebraic systems to be studied, such as BCI-algebras [1], BCK-algebras [2], BCH-algebras [3], KU-algebras [4], PSRU-algebras [5], and UP-algebras [6]. They are strongly connected with logic. For example, Imai and Iséki [1] introduced BCI-algebras in 1966 and have ties with BCI-logic being the BCI-system in combinatory logic, which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [1, 2] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The concept of KU-algebras was introduced in 2009 by Prabpayak and Leerawat [4]. In 2017, Iampan [6] introduced the concept of UP-algebras as a generalization of KU-algebras.

To overcome these uncertainties, researchers are motivated to introduce some classical theories like the theories of fuzzy sets by Zadeh in 1965 [7]. Fuzzy set is applied in many areas such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, and topology. The study of fuzzy sets is ongoing. For example, Al-Masarwah and Ahmad [8] introduced the concept of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras. Essential fuzzy ideals of rings were studied by Medhi et al. in 2008 [9]. In 2012, Pawar and Deore [10] introduced the concept of essential ideals in semirings and radical class. They generalized the concept of essential ideals of rings to semirings and established radical class of semirings closed under essential extensions. Later in 2013, Medhi and Saikia [11] studied concept T-fuzzy essential ideals and proved properties of T-fuzzy essential ideals of rings. Later in 2017,

DOI: 10.24507/icicel.16.10.1057

Wani and Pawar [12] studied essential ideals and weakly essential ideals in ternary semirings. For a commutative ring with identity R, the essential ideal graph of R is a graph whose vertex set is the set of all nonzero proper ideals of R and two vertices I and J are adjacent whenever $I + J$ is an essential ideal of R. Amjadi [13] studied the essential ideal graph of a commutative ring in 2018. In 2019, Murugadas et al. [14] studied essential ideals in near-rings using k-quasi coincidence relation. In 2020, Baupradist et al. [15] studied essential ideals and essential fuzzy ideals in semigroups. Together, they studied 0-essential ideals and 0-essential fuzzy ideals in semigroups. At present, in 2021, Gaketem and Iampan [16] introduced the concepts of essential UP-subalgebras and essential UP-ideals and the concepts of t-essential fuzzy UP-subalgebras and t-essential fuzzy UP-ideals of UP-algebras, and investigated their relationships.

This review shows that the concept of essential subsets is an important and ongoing study, but not much analysis, which has inspired the study of new concepts of essential subsets in UP-algebras. In this paper, we introduce the new concept of essential UP-filters from the concept of essential UP-ideals in UP-algebras and study some properties. We will show that the concept of essential UP-ideals is not necessarily essential UP-filters. Finally, we extend the concept of fuzzy UP-filters to t-essential fuzzy UP-filters of UP-algebras and study their relationship based on characteristic functions and upper level subsets.

2. **Preliminaries.** Now, we discussed the concept of UP-algebras and basic properties for the study of next sections.

Definition 2.1. [6] *An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:*

$$(\textit{for all } x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0), \tag{1}$$
$$(\textit{for all } x \in A)(0 \cdot x = x), \tag{2}$$
$$(\textit{for all } x \in A)(x \cdot 0 = 0), \ \textit{and} \tag{3}$$
$$(\textit{for all } x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y). \tag{4}$$

From [6], we know that the concept of UP-algebras is a generalization of KU-algebras. The binary relation $\leq$ on a UP-algebra $A = (A, \cdot, 0)$ is defined as follows:

$$(\text{for all } x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{5}$$

and the following assertions are valid (see [6, 17]).

$$(\text{for all } x \in A)(x \leq x), \tag{6}$$
$$(\text{for all } x, y, z \in A)(x \leq y, y \leq z \Rightarrow x \leq z), \tag{7}$$
$$(\text{for all } x, y, z \in A)(x \leq y \Rightarrow z \cdot x \leq z \cdot y), \tag{8}$$
$$(\text{for all } x, y, z \in A)(x \leq y \Rightarrow y \cdot z \leq x \cdot z), \tag{9}$$
$$(\text{for all } x, y, z \in A)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)), \tag{10}$$
$$(\text{for all } x, y \in A)(y \cdot x \leq x \Leftrightarrow x = y \cdot x), \tag{11}$$
$$(\text{for all } x, y \in A)(x \leq y \cdot y), \tag{12}$$
$$(\text{for all } a, x, y, z \in A)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))), \tag{13}$$
$$(\text{for all } a, x, y, z \in A)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z), \tag{14}$$
$$(\text{for all } x, y, z \in A)((x \cdot y) \cdot z \leq y \cdot z), \tag{15}$$
$$(\text{for all } x, y, z \in A)(x \leq y \Rightarrow x \leq z \cdot y), \tag{16}$$
$$(\text{for all } x, y, z \in A)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)), \text{ and} \tag{17}$$

$$(\text{for all } a, x, y, z \in A)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)). \tag{18}$$

Example 2.1. [18] *Let U be a nonempty set and let $X \in \mathcal{P}(U)$ where $\mathcal{P}(U)$ means the power set of U. Let $\mathcal{P}_X(U) = \{A \in \mathcal{P}(U) \mid X \subseteq A\}$. Define a binary operation $\triangle$ on $\mathcal{P}_X(U)$ by putting $A \triangle B = B \cap (A^C \cup X)$ for all $A, B \in \mathcal{P}_X(U)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_X(U), \triangle, X)$ is a UP-algebra. Let $\mathcal{P}^X(U) = \{A \in \mathcal{P}(U) \mid A \subseteq X\}$. Define a binary operation $\blacktriangle$ on $\mathcal{P}^X(U)$ by putting $A \blacktriangle B = B \cup (A^C \cap X)$ for all $A, B \in \mathcal{P}^X(U)$. Then $(\mathcal{P}^X(U), \blacktriangle, X)$ is a UP-algebra.*

Definition 2.2. [6, 19] *A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is called*

1) *a UP-ideal of A if*
   i) *the constant 0 of A is in S, and*
   ii) *(for all $x, y, z \in A$)$(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$,*
2) *a UP-filter of A if*
   i) *the constant 0 of A is in S, and*
   ii) *(for all $x, y \in A$)$(x \in S, x \cdot y \in S \Rightarrow y \in S)$.*

Somjanta et al. [19] proved that the concept of UP-filters is a generalization of UP-ideals.

Example 2.2. *Consider a UP-algebra $A = (A, \cdot, 0)$ where $A = \{0, 1, 2, 3\}$ is defined in the following Cayley table.*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

All the UP-ideals of A are $\{0\}$, $\{0, 2, 3\}$, and A.
All the UP-filters of A are $\{0\}$, $\{0, 1\}$, $\{0, 2, 3\}$, and A.

A *fuzzy set* ω in a nonempty set S is a function from S into the unit closed interval $[0, 1]$ of real numbers, i.e., $\omega : S \to [0, 1]$.

For any two fuzzy sets ω and ϖ in a nonempty set S, we define

1) $\omega \geq \varpi \Leftrightarrow \omega(x) \geq \varpi(x)$ for all $x \in S$.
2) $\omega = \varpi \Leftrightarrow \omega \geq \varpi$ and $\varpi \geq \omega$.
3) $(\omega \wedge \varpi)(x) = \min\{\omega(x), \varpi(x)\}$ for all $x \in S$.

If $K \subseteq S$, then the characteristic function ω_K of S is a function from S into $\{0, 1\}$ defined as follows:

$$\omega_K(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.3. *From Example 2.2, we have $\{0, 2, 3\}$ is a UP-filter of A. Then $\omega_{\{0,2,3\}}$ is a fuzzy UP-filter of A.*

Definition 2.3. [19] *A fuzzy set ω in a UP-algebra $A = (A, \cdot, 0)$ is called a fuzzy UP-filter of A if*

i) *(for all $x \in A$)$(\omega(0) \geq \omega(x))$, and*
ii) *(for all $x, y \in A$)$(\omega(y) \geq \min\{\omega(x), \omega(x \cdot y)\})$.*

We easily prove that if ω_1 and ω_2 are fuzzy UP-filters of a UP-algebra A, then $\omega_1 \wedge \omega_2$ is also a fuzzy UP-filter of A.

Theorem 2.1. [19] *Let F be a nonempty subset of a UP-algebra A. Then F is a UP-filter of A if and only if the characteristic function ω_F is a fuzzy UP-filter of A.*

Definition 2.4. *Let ω be a fuzzy set in a UP-algebra A. For any $t \in [0,1]$, the sets*

$$U(\omega;t) = \{x \in A \mid \omega(x) \geq t\} \text{ and } U^{+}(\omega;t) = \{x \in A \mid \omega(x) > t\}$$

are called an upper t-level subset and an upper t-strong level subset of ω, respectively.

Theorem 2.2. [19] *Let ω be a fuzzy set in a UP-algebra A. Then the following statements hold:*

i) ω is a fuzzy UP-filter of A if and only if for any $t \in [0,1]$, $U(\omega;t)$ is a UP-filter of A if $U(\omega;t) \neq \emptyset$;
ii) ω is a fuzzy UP-filter of A if and only if for any $t \in [0,1]$, $U^{+}(\omega;t)$ is a UP-filter of A if $U^{+}(\omega;t) \neq \emptyset$.

3. **Essential UP-filters and t-essential Fuzzy UP-filters.** In this section, we define essential UP-filter and essential fuzzy UP-filter, and together study basic properties of it.

Before we begin, let us recall the definition of an essential UP-ideal of a UP-algebra as follows.

Definition 3.1. [16] *A UP-ideal B of a UP-algebra A is called an essential UP-ideal of A if $B \cap C$ is a nonzero subset (actually, it is a nonzero UP-ideal) of A for every nonzero UP-ideal C of A. Equivalently, $\{0\} \subset B \cap C$ for every nonzero UP-ideal C of A.*

In the same way, we define an essential UP-filter of a UP-algebra as follows.

Definition 3.2. *A UP-filter F of a UP-algebra A is called an essential UP-filter of A if $\{0\} \subset F \cap E$ for every nonzero UP-filter E of A.*

Example 3.1. *By Example 2.2 we have $\{0,2,3\}$ and A are essential UP-ideals, and A is the only one essential UP-filter of A. This example shows that an essential UP-ideal is not necessarily an essential UP-filter.*

Theorem 3.1. *Let F be an essential UP-filter of A and F' be a UP-filter of A containing F. Then F' is also an essential UP-filter of A.*

Proof: Let G be a nonzero UP-filter of A. Since F is an essential UP-filter of A, we have $\{0\} \subset F \cap G \subseteq F' \cap G$. Hence, F' is an essential UP-filter of A. □

Definition 3.3. *Let $t \in [0,1)$. A fuzzy UP-filter ω of A is called a t-essential fuzzy UP-filter of A if there exists a nonzero element $x_{\varpi} \in A$ such that $t < (\omega \wedge \varpi)(x_{\varpi})$ for every nonzero fuzzy UP-filter ϖ of A.*

Theorem 3.2. *Let ω be a t-essential fuzzy UP-filter of A and ω' be a fuzzy UP-filter of A such that $\omega \leq \omega'$. Then ω' is also a t-essential fuzzy UP-filter of A.*

Proof: Let ϖ be a nonzero fuzzy UP-filter of A. Since ω is a t-essential fuzzy UP-filter of A, there exists a nonzero element $x_{\varpi} \in A$ such that $t < (\omega \wedge \varpi)(x_{\varpi}) \leq (\omega' \wedge \varpi)(x_{\varpi})$. Hence, ω' is a t-essential fuzzy UP-filter of A. □

Theorem 3.3. *Let ω be a t-essential fuzzy UP-filter of A. Then $t < \omega(0)$.*

Proof: Let ϖ be a fuzzy set in A defined by $\varpi(x) = 1$ for all $x \in A$. Then we can easily prove that ϖ is a nonzero fuzzy UP-filter of A. By assumption, there exists a nonzero element $x_{\varpi} \in A$ such that $t < (\omega \wedge \varpi)(x_{\varpi}) = \min\{\omega(x_{\varpi}), \varpi(x_{\varpi})\} = \min\{\omega(x_{\varpi}), 1\} = \omega(x_{\varpi}) \leq \omega(0)$. □

Theorem 3.4. *Let ω_1 and ω_2 be t-essential fuzzy UP-filters of A. Then $\omega_1 \wedge \omega_2$ is a t-essential fuzzy UP-filter of A.*

Proof: Let ϖ be a nonzero fuzzy UP-filter of A. Since ω_1 and ω_2 are fuzzy UP-filters of A, we have $\omega_1 \wedge \omega_2$ is a fuzzy UP-filter of A. Since ω_2 is a t-essential fuzzy UP-filter of A, we have $\omega_2 \wedge \varpi$ is a nonzero fuzzy UP-filter of A. Since ω_1 is a t-essential fuzzy UP-filter of A, there exists a nonzero element $x_\varpi \in A$ such that $t < (\omega_1 \wedge (\omega_2 \wedge \varpi))(x_\varpi) = ((\omega_1 \wedge \omega_2) \wedge \varpi)(x_\varpi)$. Hence, $\omega_1 \wedge \omega_2$ is a t-essential fuzzy UP-filter of A. $\square$

Theorem 3.5. *If F is an essential UP-filter of a nonzero UP-algebra A, then characteristic function ω_F is a 0-essential fuzzy UP-filters of A.*

Proof: Suppose that F is an essential UP-filter of A. Then F is a UP-filter of A. Thus, ω_F is a fuzzy UP-filter of A. Let ϖ be a nonzero fuzzy UP-filter of A. By Theorem 2.2 ii), we have $U^+(\varpi; 0)$ is a nonzero UP-filter of A. By assumption, there exists a nonzero element $x_\varpi \in F \cap U^+(\varpi; 0)$. Thus, $0 \leq \varpi(x_\varpi) = \min\{1, \varpi(x_\varpi)\} = \min\{\omega_F(x_\varpi), \varpi(x_\varpi)\} = (\omega_F \wedge \varpi)(x_\varpi)$. Hence, ω_F is a 0-essential fuzzy UP-filter of A. $\square$

Theorem 3.6. *Let F be a UP-filter of a nonzero UP-algebra A. If the characteristic function ω_F is a t-essential fuzzy UP-filter of A, then F is an essential UP-filter of A.*

Proof: Assume that ω_F is a t-essential fuzzy UP-filter of A. Then ω_F is a fuzzy UP-filter of A. Thus by Theorem 2.1, we have F is a UP-filter of A. Let F' be a nonzero UP-filter of A. Thus by Theorem 2.1, we have $\omega_{F'}$ is a nonzero fuzzy UP-filter of A. Since ω_F is a t-essential fuzzy UP-filter of A, there exists a nonzero element $x_{\omega_{F'}} \in A$ such that $t < (\omega_F \wedge \omega_{F'})\left(x_{\omega_{F'}}\right) = \min\left\{\omega_F\left(x_{\omega_{F'}}\right), \omega_{F'}\left(x_{\omega_{F'}}\right)\right\}$. This implies that $x_{\omega_{F'}} \in F \cap F'$, that is, $\{0\} \subset F \cap F'$. Hence, F is an essential UP-filter of A. $\square$

The following theorem shows the relationship between the upper level subset of a t-essential fuzzy UP-filter and an essential UP-filter.

Theorem 3.7. *Let ω be a fuzzy set in a nonzero UP-algebra A. Then the following statements hold:*

i) if ω is a t-essential fuzzy UP-filter of A, then $U^+(\omega; t)$ is an essential UP-filter of A;
ii) if ω is a t-essential fuzzy UP-filter of A, then $U(\omega; t)$ is an essential UP-filter of A.

Proof: i) Assume that ω is a t-essential fuzzy UP-filter of A. By Theorem 3.3, we have $0 \in U^+(\omega; t) \neq \emptyset$. Since ω is a fuzzy UP-filter of A and by Theorem 2.2 ii), we have $U^+(\omega; t)$ is a UP-filter of A. Let E be a nonzero UP-filter of A. By Theorem 2.1, we have ω_E is a nonzero fuzzy UP-filter of A. By assumption, there exists a nonzero element $x_{\omega_E} \in A$ such that $t < (\omega \wedge \omega_E)(x_{\omega_E})$. Thus, $t < \omega(x_{\omega_E})$ and $\omega_E(x_{\omega_E}) = 1$, that is, $x_{\omega_E} \in U^+(\omega; t) \cap E$. Thus, $\{0\} \subset U^+(\omega; t) \cap E$. Hence, $U^+(\omega; t)$ is an essential UP-filter of A.

ii) Assume that ω is a t-essential fuzzy UP-filter of A. By Theorem 3.3, we have $0 \in U(\omega; t) \neq \emptyset$. Since ω is a fuzzy UP-filter of A and by Theorem 2.2 i), we have $U(\omega; t)$ is a UP-filter of A. By assumption, $U^+(\omega; t)$ is an essential UP-filter of A and $U^+(\omega; t) \subseteq U(\omega; t)$. By Theorem 3.1, we have $U(\omega; t)$ is an essential UP-filter of A. $\square$

4. **Conclusion.** In this paper, we have introduced the concept of essential UP-filters from the concept of essential UP-ideals in UP-algebras and provided some properties of essential UP-filters. We have shown that the concept of essential UP-ideals is not necessarily essential UP-filters. Finally, we have studied the concept of t-essential fuzzy UP-filters of UP-algebras, which is related to the concept of essential UP-filters.

In future work, we can extend the study of essential UP-filters and t-essential fuzzy UP-filters of UP-algebras to the concept of interval-valued fuzzy sets. Further, we will study essential near UP-filters and t-essential fuzzy near UP-filters (see the definition of near UP-filters from [20], which is a generalization of UP-filters).

Acknowledgements. The authors would like to thank the anonymous referee who provided useful and detailed comments on a previous/earlier version of the manuscript.

REFERENCES

[1] Y. Imai and K. Iséki, On axiom systems of propositional calculi, XIV, *Proc. Japan Acad.*, vol.42, no.1, pp.19-22, 1966.

[2] K. Iséki, An algebra related with a propositional calculus, *Proc. Japan Acad.*, vol.42, no.1, pp.26-29, 1966.

[3] Q. P. Hu and X. Li, On BCH-algebras, *Math. Semin. Notes, Kobe Univ.*, vol.11, pp.313-320, 1983.

[4] C. Prabpayak and U. Leerawat, On ideals and congruences in KU-algebras, *Sci. Magna.*, vol.5, no.1, pp.54-57, 2009.

[5] P. Yiararyong and P. Wachirawongsakon, A new generalization of BE-algebras, *Heliyon*, vol.4, pp.1-17, 2018.

[6] A. Iampan, A new branch of the logical algebra: UP-algebras, *J. Algebra Relat. Top*, vol.5, no.1, pp.35-54, 2017.

[7] L. A. Zadeh, Fuzzy sets, *Inf. Control*, vol.8, pp.338-353, 1965.

[8] A. Al-Masarwah and A. G. Ahmad, Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras, *International Journal of Innovative Computing, Information and Control*, vol.14, no.6, pp.2025-2041, 2018.

[9] U. Medhi, K. Rajkhowa, L. K. Barthakur and H. K. Saikia, On fuzzy essential ideals of ring, *Adv. Fuzzy Sets Syst.*, vol.3, pp.287-299, 2008.

[10] K. F. Pawar and R. P. Deore, On essential ideal and radical class, *Int. J. Pure Appl. Math. Sci.*, vol.5, no.1, pp.1-5, 2012.

[11] U. Medhi and H. K. Saikia, On T-fuzzy essential ideals of rings, *J. Pure Appl. Math.*, vol.89, pp.343-353, 2013.

[12] S. Wani and K. Pawar, On essential ideals of a ternary semiring, *Sohag J. Math.*, vol.4, no.3, pp.65-68, 2017.

[13] J. Amjadi, The essential ideal graph of a commutative ring, *Asian-Eur. J. Math.*, vol.11, no.4, 1850058, 2018.

[14] P. Murugadas, V. Vetrivel and G. Manikandan, On $(\in, \in \vee q_k)$-fuzzy essential ideals of near-rings, *AIP Conf. Proc.*, vol.2177, 020046, 2019.

[15] S. Baupradist, B. Chemat, K. Palanivel and R. Chinram, Essential ideals and essential fuzzy ideals in semigroups, *J. Discrete Math. Sci. Cryptography*, vol.24, no.1, pp.223-233, 2020.

[16] T. Gaketem and A. Iampan, Essential UP-ideals and t-essential fuzzy UP-ideals of UP-algebras, *ICIC Express Letters*, vol.15, no.12, pp.1283-1289, 2021.

[17] A. Iampan, Introducing fully UP-semigroups, *Discuss. Math., Gen. Algebra Appl.*, vol.38, no.2, pp.297-306, 2018.

[18] A. Satirad, P. Mosrijai and A. Iampan, Generalized power UP-algebras, *Int. J. Math. Comput. Sci.*, vol.14, no.1, pp.17-25, 2019.

[19] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw and A. Iampan, Fuzzy sets in UP-algebras, *Ann. Fuzzy Math. Inform.*, vol.12, no.6, pp.739-756, 2016.

[20] A. Iampan, Multipliers and near UP-filters of UP-algebras, *J. Discrete Math. Sci. Cryptography*, vol.24, no.3, pp.667-680, 2021.