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INTERVAL OBSERVER FOR NONLINEAR MULTI-AGENT SYSTEMS WITH STRONG COMMUNICATION TOPOLOGY

Qingyuan Li, Yueyuan Zhang*, Jun Huang and Lei Yu

School of Mechanical and Electrical Engineering
Soochow University
No. 8, Jixue Road, Xiangcheng District, Suzhou 215137, P. R. China
lqyhenu@126.com; cauchyhot@163.com; yu_lei@suda.edu.cn
*Corresponding author: zhangyueyuan@suda.edu.cn

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Abstract. *In this article, the problem of the interval observer design for nonlinear multi-agent systems with uncertain additive disturbance is discussed. At first, a class of distributed interval observer is constructed for the nonlinear multi-agent systems, and then this paper verifies that the interval observer is able to recover the bounds of the state for each agent. Furthermore, a sufficient condition for the existence of the interval observer is formulated in the form of linear matrix inequalities. Finally, the main results are verified by an example.*

Keywords: Nonlinear multi-agent systems, Distributed interval observer, Linear matrix inequalities

1. **Introduction.** The state estimation of the system plays an important role in controller design and fault diagnosis problem. Therefore, many scholars had begun to study how to recover the state of the system, and they had proposed many methods, among which the observer is the relatively accurate one. There are many kinds of observers, such as full-order observers (FOOs) [1], reduced-order observers (ROOs) [2], and functional observers (FOs) [3]. The differences among them are that the FOs and ROOs have the lower order than FOOs, and FOs recover the linear function of state. Although these observers can recover the state of the system well, there often exist some disturbances in the actual systems, which reduces the practicability of the observers. In order to solve this problem, interval observer (IO) was proposed by Gouzé et al. [4]. An IO often contains two sub observers: one is to recover the upper bound and the other one is to recover the lower bound of the states trajectory. The coefficient matrices of the error systems are assumed to be Metzler, which can ensure the positivity of the error systems; therefore, the structure of IO is in interval form, which enables IO to estimate the range of system states. For example, based on the above assumption, Ethabet et al. [5] proposed an IO for switched systems. At present, there are two methods for IO design: one is linear matrix inequality (LMI) [6] and the other is linear programming (LP) [7]. Mazenc and Bernard [6] studied linear systems with disturbances, and used coordinate transformation to transform the system matrix into a Jordan standard form to build an IO. An IO design method for a linear system without disturbances was proposed by the LP method [7]. In recent years, IO has attracted extensive attention of scholars due to its broad application prospect. Thus, the IO is applied in many systems such as linear time-varying systems (LTVSs) [8], nonlinear systems (NSs) [9], switched systems (SSs) [10,11], and nonlinear switched systems [12]. Thabet et al. [8] constructed an IO of LTVSs by the method of time-varying coordinate transformation. The problem of IO design for NSs was discussed

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in [9], and a five-step method was proposed. In [10], an interval observer for continuous-time switching systems is constructed by coordinate transformation. A functional IO of discrete-time switched descriptor systems is designed by zonotope-based method in [11]. Che et al. [12] proposed an interval obesrver for nonlinear switched systems, and the nonlinear term satisfies the Lipschitz constraint. In addition, the combination of IOs and multi-agent systems has also received extensive attention from scholars.

Multi-agent systems (MASs) are a frontier subject of artificial intelligence [13], and compared with a single system, MASs have many advantages such as high efficiency and strong flexibility. In addition, MASs have a wide range of applications, such as formation control, trajectory planning, and robot cooperation. MASs are composed of many sub systems with the functions of calculation, execution, communication, and perception. The connection between the agents corresponds to a topological graph, and the communication topology can be divided into two categories: directed and undirected. In order to control each agent more accurately, it is necessary to study the states estimation of MASs. In [14], a distributed interval observer (DIO) was proposed for linear multi-agent systems (LMASs), and a control algorithm was designed based on this observer to realize the coordinated behavior of the LMASs. However, many practical systems are nonlinear, and it is difficult to solve the problem of nonlinear systems with the theory of linear systems. To the authors' knowledge, DIOs design for NMASs has not been adequately considered in the previous literature, which motivates this work.

Compared with the IOs, distributed interval observers (DIOs) can save computing resources better. A DIO was designed for multi-agent system in [15], which is a recent study on DIOs. This paper has the following innovations.

1) Based on the work in [14], the DIO design method for NMASs is proposed for the first time, which extends the application scope of IOs, and the monotone system theory is used to design the DIO.
2) This paper considers the uncertain disturbance of the actual system rather than the known disturbance, and only the boundary of the disturbance is known.

Based on the above discussion, this paper first constructs the framework of DIO by introducing the output feedback, and then the Lyapunov function was constructed based on the error system of DIO. At the same time, the coefficient matrices of the error systems are chosen to be Metzler. Next, the sufficient conditions for the existence of the DIO are transformed into an LMI. Finally, the output feedback gain matrix is obtained by solving the LMI. The rest of this paper is organized as follows. In Section 2, the problem statement and preliminaries are given. The main results are presented in Section 3. In Section 4, a numerical simulation example is considered to prove the effectiveness of the theory. Section 5 gives the summary and generalization.

Notations: Let matrices $M = (m_{ij})$ and $N = (n_{ij})$ have the same dimension, and if $M > N$, then $m_{ij} > n_{ij}$. $Z > 0$ (≥ 0) represents a (semi-) positive definite matrix Z, and $Z < 0$ (≤ 0) represents a (semi-) negative definite matrix Z. $\mathbb{G} = \{\mathbb{V}, \mathbb{E}\}$ is used to represent the relationship between each agent, in which $\mathbb{V} = \{1, 2, \ldots, N\}$ represents each agent and $\mathbb{E} = \{(i, j)$ – if there is a connection between agent i and agent $j\}$, and the $\mathcal{A} = (a_{ij}) \in \mathbf{R}^{N\times N}$ is used to represent the adjacency matrix of $\mathbb{G}$. λ is the eigenvalues. $\otimes$ represents the Kronecker product.

2. **Problem Statement and Preliminaries.**

The state-space equation of each agent is shown as follows:

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i + f(x_i) + \omega_i, \\ y_i = \overline{C}x_i, \end{cases} \tag{1}$$

where $i = \{1, 2, \ldots, N\}$, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and $\overline{C} \in \mathbf{R}^{p\times n}$ are the constant matrices, respectively. $x_i \in \mathbf{R}^n$ is the state vector. $u_i \in \mathbf{R}^m$ is the input vector, and $y_i \in \mathbf{R}^p$ is the output vector. $\omega_i \in \mathbf{R}^n$ represents the disturbance of the ith agent. $f(x_i)$ denotes the nonlinear dynamics of the ith agent and $f(x_i) \in \mathbf{R}^n$. The disturbance, the nonlinear dynamics and the initial states are bounded as follows:

$$\underline{\omega}_i(t) \leq \omega_i(t) \leq \overline{\omega}_i(t), \tag{2}$$

$$f\left(\underline{x}_i\right) \leq f(x_i) \leq f\left(\overline{x}_i\right), \tag{3}$$

where $f(x_i)$ satisfies

$$|f(a) - f(b)| < \gamma|b - a|.$$

γ is a constant, and

$$\underline{x}_i(0) \leq x_i(0) \leq \overline{x}_i(0). \tag{4}$$

The element in matrix $\mathcal{A}$ is

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathbb{E} \\ 0, & \text{otherwise} \end{cases}, \tag{5}$$

and use $L = (l_{ij}) \in \mathbf{R}^{N\times N}$ to represent the Laplacian matrix of $\mathbb{G}$,

$$l_{ij} = \begin{cases} -a_{ij}, & \text{if } i \neq j \\ \sum\limits_{j\in N(i)} a_{ij}, & \text{if } i = j \end{cases}. \tag{6}$$

The eigenvalues of L are set to λ_i, and then the sequences $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ can be obtained by sorting λ_i in ascending order. In addition, if $\mathbb{G}$ is connected, $\lambda_2 > 0$.

2.1. **Definition.** In order to more intuitively express the upper and lower bounds of the states trajectory of system (1), this paper gives two sub observers shown as follows.

$$\begin{cases} \dot{\underline{x}}_i(t) = \underline{g}\left(\underline{x}_i(t), u_i(t), f(\underline{x}_i), \underline{\omega}_i(t), y_i, y_j, \underline{y}_i, \underline{y}_j\right), & j \in N(i) \\ \dot{\overline{x}}_i(t) = \overline{g}\left(\overline{x}_i(t), u_i(t), f(\overline{x}_i), \overline{\omega}_i(t), y_i, y_j, \overline{y}_i, \overline{y}_j\right), & j \in N(i) \end{cases}. \tag{7}$$

(7) is a DIO of (1) when (7) satisfies the following conditions.

1) If $\underline{\omega}_i(t) = \overline{\omega}_i(t) = 0$, then as time goes on, $\underline{x}_i(t)$ will equal $\overline{x}_i(t)$.
2) The solutions of $(\underline{x}_i(t), \overline{x}_i(t))$ are defined over $[0, +\infty)$, for all continuous functions $\underline{\omega}_i(t)$ and $\overline{\omega}_i(t)$.
3) From the moment the system starts, it needs to satisfy $\overline{x}_i(t) - x_i(t) \geq 0$, and $x_i(t) - \underline{x}_i(t) \geq 0$.

The following Equation (8) is the internal structure of the DIO:

$$\begin{cases} \dot{\overline{x}}_i = A\overline{x}_i + Bu_i + f(\overline{x}_i) + \overline{\omega}_i(t) + gK\left[\sum\limits_{j\in N(i)} a_{ij}(y_i - y_j) - \sum\limits_{j\in N(i)} a_{ij}\left(\overline{y}_i - \overline{y}_j\right)\right], \\ \dot{\underline{x}}_i = A\underline{x}_i + Bu_i + f(\underline{x}_i) + \underline{\omega}_i(t) + gK\left[\sum\limits_{j\in N(i)} a_{ij}(y_i - y_j) - \sum\limits_{j\in N(i)} a_{ij}\left(\underline{y}_i - \underline{y}_j\right)\right], \end{cases} \tag{8}$$

where K is the feedback gain matrix between the adjacent agents, and g is the couple gain.

2.2. **Lemma.** Lemma 2.1 and Lemma 2.2 are used in the following analysis of this paper.

Lemma 2.1. [16] *If $\mathbb{G}$ is a strongly connected topology and $\phi = [\phi_1, \ldots, \phi_N]^T$ is the left positive eigenvector of L related to 0 eigenvalue, then $\Phi L + L^T\Phi \geq 0$, where $\Phi = diag(\phi_1, \ldots, \phi_N)$.*

Lemma 2.2. [16] *Based on Lemma 2.1, the generalized algebraic connectivity of* L *is defined as* $\beta(L) = \min_{\phi^T x=0, x\neq 0} \frac{x^T(\Phi L + L^T\Phi)x}{2x^T\Phi x} \geq 0$. *For balanced graphs,* $\beta(L) = \lambda_2$.

3. **Main Result.** The main results of this paper are shown below:

Theorem 3.1. *If there exists a matrix* $Q \in \mathbf{R}^{n\times n}$, *such that*

$$QA + A^TQ - \overline{C}^T\overline{C} + Q^2 + \gamma^2 I_n < 0, \tag{9}$$

and $I_N \otimes A - 2cL \otimes K\overline{C}$ *is Metzler, then (7) is a stable DIO of system (1), where* γ *is the Lipschitz constant, and* $g \geq 1/\lambda_2(L)$, *furthermore* $K = \frac{1}{2}Q^{-1}\overline{C}^T$.

Proof: From (8) we can get

$$\begin{cases} \dot{\overline{x}} = \left(I_N \otimes A - cL \otimes K\overline{C}\right)\overline{x} + (I_N \otimes B)u + f\left(\overline{x}\right) + \overline{\omega} + g[L \otimes K]y, \\ \dot{\underline{x}} = \left(I_N \otimes A - cL \otimes K\overline{C}\right)\underline{x} + (I_N \otimes B)u + f\left(\underline{x}\right) + \underline{\omega} + g[L \otimes K]y. \end{cases} \tag{10}$$

We set $\overline{e} = \overline{x} - x$ and $\underline{e} = x - \underline{x}$; thus, we can get

$$\begin{cases} \dot{\overline{e}} = \left(I_N \otimes A - L \otimes gK\overline{C}\right)\overline{e} + \Delta\overline{f} + \Delta\overline{\omega}, \\ \dot{\underline{e}} = \left(I_N \otimes A - L \otimes gK\overline{C}\right)\underline{e} + \Delta\underline{f} + \Delta\underline{\omega}. \end{cases} \tag{11}$$

If $I_N \otimes A - L \otimes gK\overline{C}$ is Metzler matrix, the errors $\overline{e} > 0$ and $\underline{e} > 0$, so the condition $\underline{x}_i(t) \leq x_i(t) \leq \overline{x}_i(t)$ will be satisfied.

Next, choose the following Lyapunov function:

$$\begin{aligned} V &= V_1 + V_2, \\ V_1 &= \overline{e}^T(I_N \otimes Q)\overline{e}, \\ V_2 &= \underline{e}^T(I_N \otimes Q)\underline{e}. \end{aligned} \tag{12}$$

The $\dot{V}_1$ and $\dot{V}_2$ will be gotten from (12):

$$\begin{aligned} \dot{V} &= \dot{V}_1 + \dot{V}_2 \\ &= 2\overline{e}^T(I_N \otimes Q)\dot{\overline{e}} + 2\underline{e}^T(I_N \otimes Q)\dot{\underline{e}} \\ &= 2\overline{e}^T(I_N \otimes Q)\left[(I_N \otimes A) - \left(L \otimes gK\overline{C}\right)\right]\overline{e} + 2\overline{e}^T(I_N \otimes Q)\Delta\overline{f} + 2\overline{e}^T(I_N \otimes Q)\Delta\overline{\omega} \\ &\quad + 2\underline{e}^T(I_N \otimes Q)\left[(I_N \otimes A) - \left(L \otimes gK\overline{C}\right)\right]\underline{e} + 2\underline{e}^T(I_N \otimes Q)\Delta\underline{f} + 2\underline{e}^T(I_N \otimes Q)\Delta\underline{\omega}. \end{aligned} \tag{13}$$

Because of $\overline{\omega}(t) = \underline{\omega}(t) = 0$, then $\omega(t) = 0$, $2\underline{e}^T(I_N \otimes Q)\Delta\underline{\omega} = 0$, and $2\overline{e}^T(I_N \otimes Q)\Delta\overline{\omega} = 0$, the first item in (13) can be written as

$$2\overline{e}^T(I_N \otimes Q)(I_N \otimes A)\overline{e} = \overline{e}^T\left(I_N \otimes \left(QA + A^TQ\right)\right)\overline{e}. \tag{14}$$

By $K = \frac{1}{2}Q^{-1}\overline{C}^T$ the second item in (13) can be written as

$$\begin{aligned} -2\overline{e}^T(I_N \otimes Q)\left(L \otimes gK\overline{C}\right)\overline{e} &= -2\overline{e}^T(I_N \otimes Q)\frac{1}{2}\left(L \otimes gQ^{-1}\overline{C}^T\overline{C}\right)\overline{e} \\ &= -\overline{e}^T\left(gL \otimes \overline{C}^T\overline{C}\right)\overline{e} \\ &= -\overline{e}^T\left(g\frac{L + L^T}{2} \otimes \overline{C}^T\overline{C}\right)\overline{e}, \end{aligned} \tag{15}$$

and then, select the coupling strength $g \geq 1/\beta(L) = 1/\lambda_2$, which is defined in Lemma 2.2.

Because L is real symmetric, there exists a real orthogonal matrix W, such that

$$L = W^T\Theta W, \tag{16}$$

where $\Theta = diag\{0, \lambda_2, \lambda_3, \ldots, \lambda_N\}$, we denoted $\xi = (W \otimes I_n)\overline{e}$ and then Equation (15) can be written as

$$\begin{aligned}
-\overline{e}^T\left(g\frac{L+L^T}{2}\otimes\overline{C}^T\overline{C}\right)\overline{e} &= \xi^T\left[(-g\Theta)\otimes\overline{C}^T\overline{C}\right]\xi \\
&= \sum_{i=2}^{N}(-g\lambda_i)\xi_i^T\overline{C}^T\overline{C}\xi_i \\
&\leq -\sum_{i=2}^{N}\xi_i^T\overline{C}^T\overline{C}\xi_i \\
&= \overline{e}^T\left(-I_N\otimes\overline{C}^T\overline{C}\right)\overline{e}. \qquad (17)
\end{aligned}$$

The third item in (13) can be written as

$$2\overline{e}^T(I_N\otimes Q)\Delta\overline{f}\leq 2\overline{e}^T(I_N\otimes Q)(I_N\otimes Q)\overline{e}+\gamma^2\overline{e}^T\overline{e}. \qquad (18)$$

By bringing (14), (17) and (18) into $\dot{V}_1$,

$$\dot{V}_1\leq\overline{e}^T\left\{I_N\otimes\left[QA+A^TQ-\overline{C}^T\overline{C}+Q^2+\gamma^2 I_n\right]\right\}\overline{e}. \qquad (19)$$

The following proof method for $\dot{V}_2$ is similar to the previous one, so the condition (20) will be gotten:

$$QA+A^TQ-\overline{C}^T\overline{C}+Q^2+\gamma^2 I_n<0. \qquad (20)$$

Using Schure complement to solve Q, by the proof we can obtain that the errors are convergent. (10) are an exponential IO of system (1).

4. **Numerical Example.** Consider the system (1) with

$$A=\begin{bmatrix}-5 & 4\\ 4 & -6\end{bmatrix},\quad B=\begin{bmatrix}1\\1\end{bmatrix},\quad \overline{C}=\begin{bmatrix}0.01 & 0.01\end{bmatrix}.$$

The upper and lower bounds of ω_i, nonlinear term and coupling gain are as follows:

$$-0.01\leq\omega_i\leq 0.01,\quad f(x)=0.1(x-\sin x),\quad g=5.$$

The matrices Q and K can be obtained by solving the LMI in (20):

$$Q=\begin{bmatrix}0.227 & 0.1487\\ 0.1487 & 0.1898\end{bmatrix},\quad K=\begin{bmatrix}0.0098\\ 0.0186\end{bmatrix}.$$

Topology is shown in Figure 1, in which the number represents the agent, which can communicate with each other. The system considered in this paper is two-dimensional, so each agent has two figures. The simulation results of agent 1 are shown in Figure 2 and Figure 3. The simulation results of agent 2 are shown in Figure 4 and Figure 5, and the simulation results of agent 3 are shown in Figure 6 and Figure 7. Each figure from 2 to 7 shows a state trajectory and the corresponding DIO, where the solid line represents the

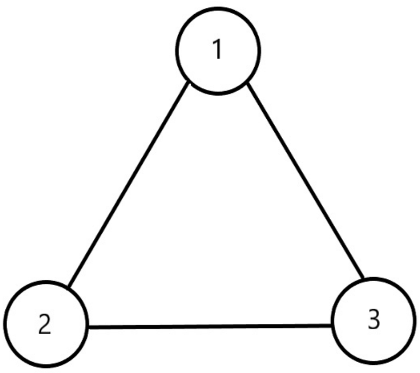

FIGURE 1. Topology of multi-agent

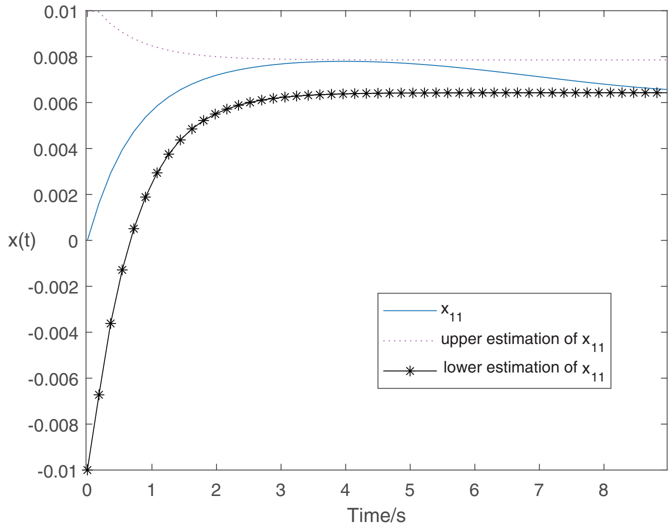

FIGURE 2. Evolution of the real state $x_{11}(t)$ and the estimations $\overline{x}_{11}(t)$, $\underline{x}_{11}(t)$

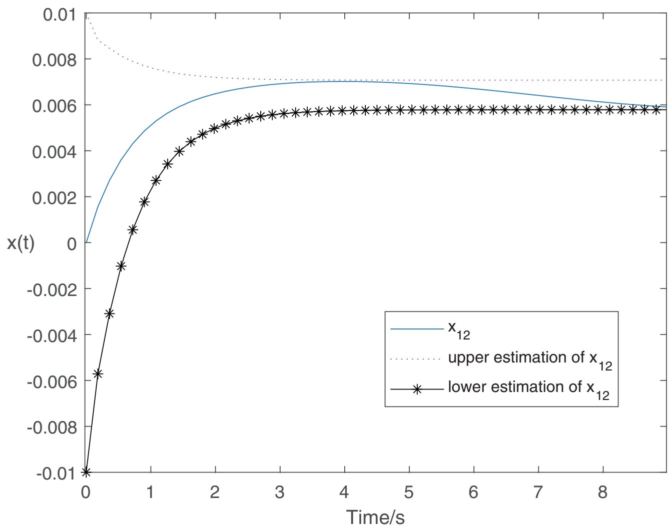

FIGURE 3. Evolution of the real state $x_{12}(t)$ and the estimations $\overline{x}_{12}(t)$, $\underline{x}_{12}(t)$

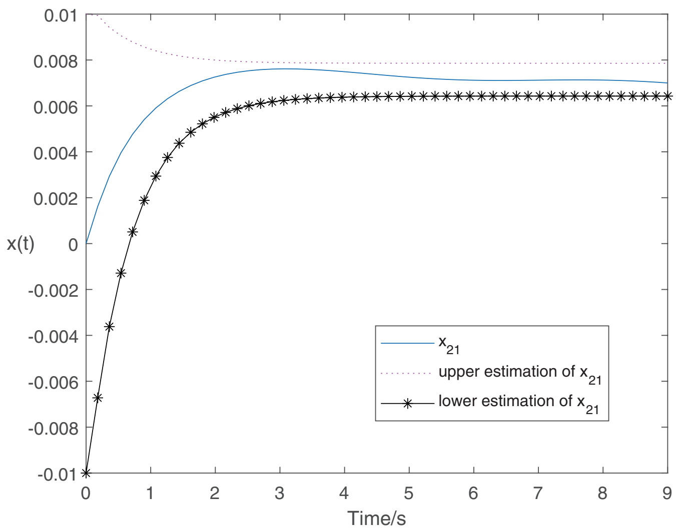

FIGURE 4. Evolution of the real state $x_{21}(t)$ and the estimations $\overline{x}_{21}(t)$, $\underline{x}_{21}(t)$

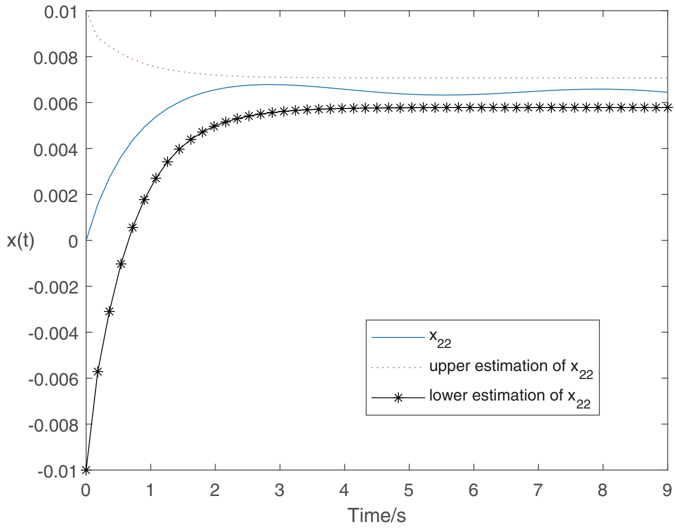

FIGURE 5. Evolution of the real state $x_{22}(t)$ and the estimations $\overline{x}_{22}(t)$, $\underline{x}_{22}(t)$

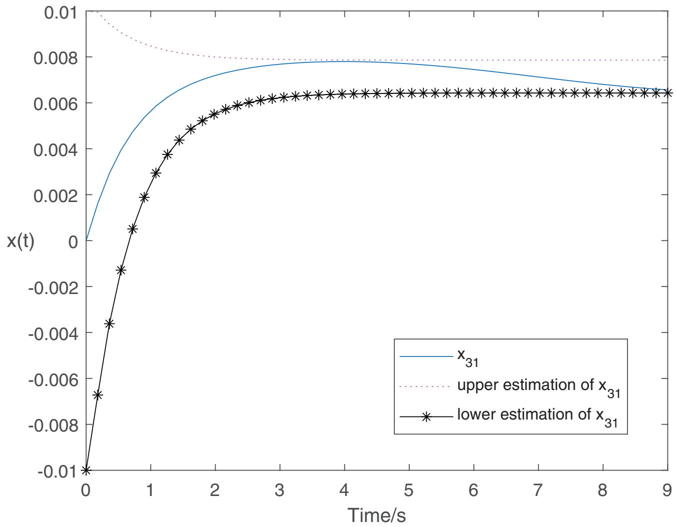

FIGURE 6. Evolution of the real state $x_{31}(t)$ and the estimations $\overline{x}_{31}(t)$, $\underline{x}_{31}(t)$

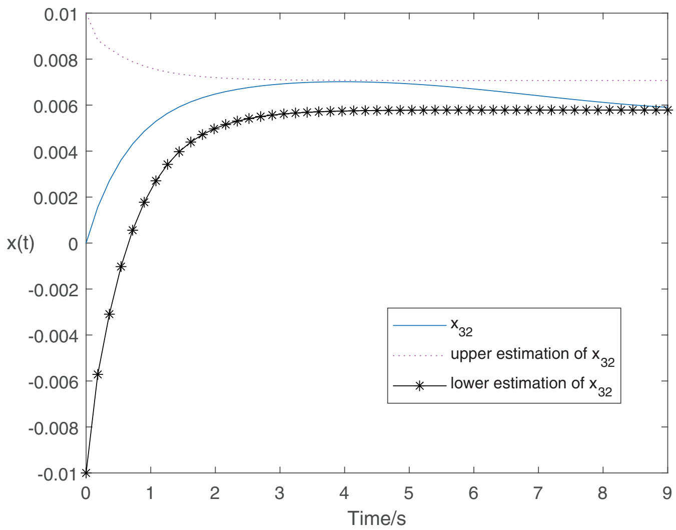

FIGURE 7. Evolution of the real state $x_{32}(t)$ and the estimations $\overline{x}_{32}(t)$, $\underline{x}_{32}(t)$

state trajectory, and the dotted line and asterisk represent the upper and lower bounds of DIO, respectively. We can see that the state of the system is accurately estimated. The simulation outcome shows that the observer which is proposed in the paper is effective.

5. **Conclusions.** In this paper, the problem of DIO design for NMASs with uncertain additive disturbance is discussed. The DIO design method for NMASs is proposed for the first time, which extends the application scope of IOs, and the monotone system theory is used to design the DIO. Finally, the simulation results show that the DIO proposed in this paper is feasiable. Future work includes DIO design for NMASs with switched topologyies rather than a fixed topology.

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