ICIC Express Letters
Volume **16**, Number **11**, November **2022**

ICIC International ©2022 ISSN 1881-803X
pp. 1223–1227

INTUITIONISTIC FUZZY UNIFORM MODULES

Samruam Baupradist[1,3,*] and Hoang Dinh Hai[2]

[1]Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Phayathai Road, Pathumwan, Bangkok 10330, Thailand
[*]Corresponding author: samruam.b@chula.ac.th

[2]Faculty of Natural Sciences
Hong Duc University
565 Quang Trung, Dong Ve, Thanh Hoa 440000, Vietnam
hoangdinhhai@hdu.edu.vn

[3]Centre of Excellence in Mathematics
CHE, Si Ayuthaya Road, Bangkok 10400, Thailand

Received January 2022; accepted April 2022

Abstract. *In this paper, we introduce the notion of intuitionistic fuzzy uniform modules. A right R-module $M \neq \{0\}$ is called an intuitionistic fuzzy uniform module, if for any $A = (\mu_A, \nu_A) \in IFM(M)$ with $A_* \neq M$; A is an intuitionistic fuzzy essential submodule of M where $A_* = \{x \in M \mid \mu_A(x) = 1$ and $\nu_A(x) = 0\}$. We obtain various properties of intuitionistic fuzzy uniform modules and establish necessary and sufficient conditions for intuitionistic fuzzy uniform modules.*

Keywords: Essential submodules, Uniform modules, Intuitionistic fuzzy essential submodules, Intuitionistic fuzzy uniform modules

1. **Introduction and Preliminaries.** The notion of intuitionistic fuzzy set was introduced by Atanassov [1] in 1986. Intuitionistic fuzzy set is used to mathematically deal with imprecise and uncertain conditions. Since then these ideas have been applied to other algebraic structures like semigroups, groups, rings, and modules. In 1989, Biswas [3] applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of group. Hur et al. [6] applied this concept to rings and introduced the notion of intuitionistic fuzzy subrings and ideals of a ring. Davvaz et al. in [4] introduced the notion of intuitionistic fuzzy submodules of module as an extension of fuzzy submodules of module introduced by Negoita and Ralescu [9]. Consequently, intuitionistic fuzzy module generated by intuitionistic fuzzy sets [8], intuitionistic fuzzy quotient modules [7, 10], residual quotient and annihilator of intuitionistic fuzzy sets of ring and module [13] were investigated. Basnet [2] defined intuitionistic fuzzy essential submodules and investigated various characteristics of such submodule. Authors in [12] defined intuitionistic fuzzy superfluous submodules and that of intuitionistic fuzzy hollow submodules in [14]. For more information about intuitionistic fuzzy modules see [11, 15, 16].

In this paper, we define the notion of intuitionistic fuzzy uniform modules and discuss its properties. A relationship between uniform module and the intuitionistic fuzzy uniform submodule has been obtained.

Throughout of this paper, all rings are associative with identity $1 \neq 0$ and all modules are unitary right R-modules and otherwise stated.

DOI: 10.24507/icicel.16.11.1223

Definition 1.1 ([17]). *A submodule N of a right R-module M is called an essential (or large) in M, if every non-zero submodule A of M, $N \cap A \neq \{0\}$. Then M is called an essential extension of N and we write $N \subset^*_> M$.*

Definition 1.2 ([5]). *A non-zero right R-module M is said to be uniform, if any two non-zero submodules of M have non-zero intersection, i.e., every non-zero submodule is an essential in M.*

Definition 1.3 ([1]). *Let X be a non-empty set. An intuitionistic fuzzy set (IFS) A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu(x)\rangle \mid x \in X\}$, where the function $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$.*

The class of intuitionistic fuzzy subsets of X is denoted by $\mathit{IFS}(X)$.

Remark 1.1. *We denote the IFS $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ by $A = (\mu_A, \nu_A)$.*

Definition 1.4 ([1]). *Let A and B be IFSs of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in X\}$. Then,*

1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.
2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$.
4) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$.
5) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$.

Definition 1.5 ([7]). *Let M be a right R-module. An IFS $A = (\mu_A, \nu_A)$ of M is called an intuitionistic fuzzy (right) submodule (IFSM), if*

1) $\mu_A(0) = 1$ and $\nu_A(0) = 0$,
2) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$ for all $x, y \in M$,
3) $\mu_A(xr) \geq \mu_A(x)$ and $\nu_A(xr) \leq \nu_A(x)$ for all $x \in M$, $r \in R$.

We denote the set of all IFSM of a right R-module M as $\mathit{IFSM}(M)$.

Definition 1.6 ([7]). *If $A = (\mu_A, \nu_A)$ is an IFSM of a right R-module M, then we denote*

$$A_* = \{x \in M \mid \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}.$$

Remark 1.2. *A_* is a submodule of a right R-module M.*

Definition 1.7 ([2]). *We define two IFS Ω and $\Omega(M)$ of right R-module M as*

$$\Omega(x) = \begin{cases} (1,0) & \text{if } x = 0, \\ (0,1) & \text{otherwise} \end{cases}$$

for all $x \in M$ and

$$\Omega(M)(x) = (1,0) \text{ for all } x \in M.$$

Remark 1.3. *The IFS Ω and $\Omega(M)$ are IFSMs of M where M is a right R-module.*

2. **Intuitionistic Fuzzy Uniform Modules.**

In this section, we would like to discuss and prove our main results.

Definition 2.1 ([2]). *An IFSM A ($A \neq \Omega$) of a right R-module M is said to be an intuitionistic fuzzy essential submodule (IFESM) of M (or of $\Omega(M)$), if for any IFSM B of M, $A \cap B = \Omega \Rightarrow B = \Omega$ or equivalently, $B \neq \Omega \Rightarrow A \cap B \neq \Omega$.*

Remark 2.1. *We denote it by $A \subset^*_{>IF} \Omega(M)$ or $A \subset^*_{>IF} M$.*

Theorem 2.1. *Let A be an IFSM of a right R-module M. Then, A is an intuitionistic fuzzy essential submodule of $\Omega(M)$ if and only if A_* is an essential submodule of M.*

Proof: Suppose that A is an intuitionistic fuzzy essential submodule of $\Omega(M)$. Let N be a submodule of M such that $A_* \cap N = \{0\}$ and χ_N the characteristic intuitionistic function of N defined by $\chi_N(x) = (\mu_{\chi_N}(x), \nu_N(x))$, where

$$\mu_{\chi_N}(x) = \begin{cases} 1 & \text{if } x \in N, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in M$ and

$$\nu_{\chi_N}(x) = \begin{cases} 0 & \text{if } x \in N, \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in M$. Then χ_N is an IFSM of M such that $(\chi_N)_* = N$. Since $\{0\} = A_* \cap N = A_* \cap (\chi_N)_* = (A \cap \chi_N)_*$, $A \cap \chi_N = \Omega$. However, $A \subset^*_{>IF} \Omega(M)$, so $\chi_N = \Omega$. Thus, $N = (\chi_N)_* = (\Omega)_* = \{0\}$ and hence, $A \subset^*_{>} M$. Conversely, suppose that A_* is an essential submodule of M. Let B be any IFSM of M such that $A \cap B = \Omega$. Then $A_* \cap B_* = (A \cap B)_* = (\Omega)_* = \{0\}$ but A_* is an essential submodule of M, so $B_* = \{0\}$ and thus $B = \Omega$. Hence, A is an intuitionistic fuzzy essential submodule of $\Omega(M)$. □

Proposition 2.1. *Let A and B be IFSMs of a right R-module M. If $A \subset^*_{>IF} \Omega(M)$ or $B \subset^*_{>IF} \Omega(M)$, then $(A + B) \subset^*_{>IF} \Omega(M)$.*

Proof: Suppose that $A \subset^*_{>IF} \Omega(M)$ or $B \subset^*_{>IF} \Omega(M)$.

Case 1. $A \subset^*_{>IF} \Omega(M)$. Let C be IFSM of M such that $(A + B) \cap C = \Omega$. Since $\Omega \subseteq A \cap C \subseteq (A + B) \cap C = \Omega$, $A \cap C = \Omega$. However, $A \subset^*_{>IF} \Omega(M)$, $C = \Omega$. Hence, $(A + B) \subset^*_{>IF} \Omega(M)$.

Case 2. $B \subset^*_{>IF} \Omega(M)$. Similarly, we can show that $(A + B) \subset^*_{>IF} \Omega(M)$. □

Corollary 2.1. *Let A be an IFSM of a right R-module M and $\phi : M \longrightarrow N$ an isomorphism. If $A \subset^*_{>IF} \Omega(N)$, then $\phi^{-1}(A) \subset^*_{>IF} \Omega(M)$.*

Proof: Suppose that $A \subset^*_{>IF} \Omega(N)$. By Theorem 2.1, $A_* \subset^*_{>} N$ and thus $\phi^{-1}(A_*) \subset^*_{>} M$ but $\phi^{-1}(A)_* = \phi^{-1}(A_*)$ thus $\phi^{-1}(A)_* \subset^*_{>} M$ so $\phi^{-1}(A) \subset^*_{>IF} \Omega(M)$. □

Corollary 2.2. *Let A and B be IFSMs of a right R-module M such that $A \subseteq B$. If $A \subset^*_{>IF} \Omega(M)$, then $B \subset^*_{>IF} \Omega(M)$.*

Proof: Suppose that $A \subset^*_{>IF} \Omega(M)$. By Theorem 2.1, $A_* \subset^*_{>} M$. Since $A \subseteq B$, $A_* \subseteq B_*$ but $A_* \subset^*_{>} M$. So $B_* \subset^*_{>} M$. By Theorem 2.1, $B \subset^*_{>IF} \Omega(M)$. □

We define the definition of intuitionistic fuzzy uniform submodule of a right R-module M as follows.

Definition 2.2. *An intuitionistic fuzzy submodule B of a right R-module M with $B_* \neq \{0\}$ is said to be an intuitionistic fuzzy uniform submodule, if for any $A \in IFM(M)$ with $A \subsetneq B$ and $A_* \neq B_*$, A is an intuitionistic fuzzy essential submodule of B. Also, a right R-module $M \neq \{0\}$ is called an intuitionistic fuzzy uniform module, if for any $A \in IFM(M)$ with $A_* \neq M$, A is an intuitionistic fuzzy essential submodule of M.*

Theorem 2.2. *Let M be a right R-module. Then, M is a uniform module if and only if M is an intuitionistic fuzzy uniform module.*

Proof: Suppose that M is a uniform module. Let $A \in IFM(M)$ such that $A_* \neq M$. Since $\{0\} \neq A_* \subset_{>} M$ and M is a uniform module, A_* is an essential submodule of M. By Theorem 2.1, A is an intuitionistic fuzzy essential submodule of M. Hence, M is an intuitionistic fuzzy uniform module. Conversely, suppose that M is an intuitionistic fuzzy uniform module. Let N be non-zero proper submodule of M and χ_N the characteristic intuitionistic function of N defined by $\chi_N(x) = (\mu_{\chi_N}(x), \nu_N(x))$, where

$$\mu_{\chi_N}(x) = \begin{cases} 1 & \text{if } x \in N, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in M$ and

$$\nu_{\chi_N}(x) = \begin{cases} 0 & \text{if } x \in N, \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in M$. Then $\chi_N \neq \chi_M$ but M is an intuitionistic fuzzy uniform module, so χ_N is an intuitionistic fuzzy essential submodule of M. By Theorem 2.1, $N = (\chi_N)_*$ is an essential submodule of M. Hence, M is a uniform module. □

Theorem 2.3. *Let $B \in IFM(M)$ such that $B \neq \chi_{\{0\}}$. Then, B is an intuitionistic fuzzy uniform submodule of M if and only if B_* is a uniform submodule of M.*

Proof: Suppose that B is an intuitionistic fuzzy uniform submodule of M. Let N be a non-zero proper submodule of B_*. Then $\chi_N \subsetneq B$ where χ_N is the characteristic intuitionistic function of N but B is an intuitionistic fuzzy uniform submodule of M, χ_N is an intuitionistic fuzzy essential submodule of M. By Theorem 2.1, $N = (\chi_N)_*$ is an essential submodule of B_*. Hence, B_* is a uniform submodule of M. Conversely, suppose that B_* is a uniform submodule of M. Let $A \in IFM(M)$ such that $A \subseteq B$ and $A_* \neq B_*$, then $A_* \subsetneq B_*$ but B_* is a uniform submodule of M, A_* is an essential submodule of M. By Theorem 2.1, A is an intuitionistic fuzzy essential submodule of B. Hence, B is an intuitionistic fuzzy uniform submodule of M. □

Recall that an IFSM $\chi_{\{0\}} \neq A$ of M is said to be *indecomposable* IFSM, if there does not exist IFSM B and $C(\neq \chi_{\{0\}}, A)$ of M such that $A = B \oplus C$.

Theorem 2.4. *Every intuitionistic fuzzy uniform submodule is indecomposable.*

Proof: Let A be an intuitionistic fuzzy uniform submodule of a right R-module M. Suppose that A is not indecomposable. There exist $B, C \in IFM(M)$ such that $A = B \oplus C$ and $B_*, C_* \neq \chi_{\{0\}}$. Since A is an intuitionistic fuzzy uniform submodule of M, $B \cap C \neq \chi_{\{0\}}$. However, $B \oplus C$, $B \cap C = \chi_{\{0\}}$, it is contradiction. Therefore, A is indecomposable. □

Corollary 2.3. *If M is an intuitionistic fuzzy uniform module, then χ_M is indecomposable.*

3. **Conclusion.** In this paper, we have defined intuitionistic fuzzy uniform submodule of right R-module and some of their properties were investigated. The research was able to distinguish the properties between uniform modules and intuitionistic fuzzy uniform modules. In our further study, we can investigate various spanning dimensions of intuitionistic fuzzy uniform modules.

Acknowledgment. The authors are very grateful to the referees for many valuable comments and suggestions which helped to improve the paper. We were supported from Center of Excellence in Mathematics, CHE, Si Ayuthaya Road, Bangkok 10400, Thailand.

REFERENCES

[1] K. T. Atanassov, Intuitionistic fuzzy sets, *Sets and Systems*, vol.20, no.1, pp.87-96, 1986.
[2] D. K. Basnet, *Topics in Intuitionistic Fuzzy Algebra*, Lambert Academic Publishing, Germany, 2011.
[3] R. Biswas, Intuitionistic fuzzy subgroups, *Mathematical Forum*, vol.10, pp.39-44, 1989.
[4] B. Davvaz, W. A. Dudek and Y. B. Jun, Intuitionistic fuzzy H_v-submodules, *Information Sciences*, vol.176, pp.285-300, 2006.
[5] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Extending modules, *Pitman Research Notes in Mathematics No.313*, Longman Scientific and Technical, 1994.
[6] K. Hur, H. W. Kang and H. K. Song, Intuitionistic fuzzy subgroups and subrings, *Honam Math J.*, vol.25, no.1, pp.19-41, 2003.
[7] P. Isaac and P. P. John, On intuitionistic fuzzy submodules of a module, *Int. J. of Mathematical Sciences and Applications*, vol.1, no.3, pp.1447-1454, 2011.

[8] P. P. John and P. Isaac, IFSM's of an R-module – A study, *International Mathematical Forum*, vol.19, no.7, pp.935-943, 2012.
[9] C. V. Negoita and D. A. Ralescu, *Applications of Fuzzy Sets and Systems Analysis*, Birkhauser, Basel, 1975.
[10] S. Rahman and H. K. Saikia, Some aspects of Atanassov's intuitionistic fuzzy submodules, *Int. J. Pure and Appl. Mathematics*, vol.77, no.3, pp.369-383, 2012.
[11] P. K. Sharma, Chandni and N. Bhardwa, Category of intuitionistic fuzzy modules, *Mathematics*, vol.10, no.3, p.399, 2022.
[12] P. K. Sharma and G. Kaur, Intuitionistic fuzzy superfluous submodule, *The 20th ICIFS, Notes on Intuitionistic Fuzzy Sets*, vol.22, no.3, pp.34-46, 2016.
[13] P. K. Sharma and G. Kaur, Residual quotient and annihilator of intuitionistic fuzzy sets of ring and module, *International Journal of Computer Science and Information Technology*, vol.9, no.4, pp.1-15, 2017.
[14] P. K. Sharma and G. Kaur, Intuitionistic fuzzy hollow submodules, *Notes on Intuitionistic Fuzzy Sets*, vol.24, no.2, pp.25-32, 2018.
[15] P. K. Sharma, Kanchan and D. S. Pathania, On decomposition of intuitionistic fuzzy prime submodules, *Notes on Intuitionistic Fuzzy Sets*, vol.26, no.2, pp.25-32, 2020.
[16] B. Talaee, Intuitionistic fuzzy small submodules and their properties, *Fuzzy Information and Engineering*, vol.11, no.3, pp.307-319, 2019.
[17] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, London, Tokyo, 1991.