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# ON PURE FUZZY IDEALS IN ORDERED TERNARY SEMIGROUPS

Jatuporn Sanborisoot
Algebra and Applications Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Khamriang Sub-District, Kantarawichai District, Mahasarakham 44150, Thailand
jatuporn.san@msu.ac.th

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Abstract. *In this paper, we introduce the concepts of pure fuzzy and weakly pure fuzzy ideals in ordered ternary semigroups. Characterizations of right weakly regular ordered ternary semigroups by fuzzy ideals are investigated. Moreover, we characterize those ordered ternary semigroups for which each fuzzy ideal is weakly pure.*
**Keywords:** Ordered ternary semigroup, Pure fuzzy ideal, Right weakly regular

1. **Introduction.** In 1932, the literature of ternary algebraic system was introduced by Lehmer in [1]. The notion of ternary semigroup was first introduced by Banach who showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroup (cf. [2, 6]). Sioson developed ideal theory of ternary semigroups in [3]. In [4], the theory of partially ordered semigroups was defined by Kehayopulu. Later, Iampan introduced ordered ternary semigroup and characterized the minimality and maximality of ordered lateral ideals in ordered ternary semigroups in [5]. In [6], Sanborisoot and Changphas defined the notions of pure ideals of ordered ternary; moreover, the authors characterized a left weakly regular ordered semigroup and a right weakly regular ordered semigroup in terms of pure ideals.

The fuzzy set was initiated by Zadeh in [7]. Later, many papers on fuzzy sets showed importance of the concept and applications to logic, set theory, group theory, groupoids, real analysis, measure theory, topology, etc. The study of fuzzy algebraic strutures started with the introduction of the concepts of fuzzy subgroup in the pioneering Rosenfeld's in [8]. The fuzzy left (right) ideals in semigroups were introduced and studied by Kuroki in [9, 10]. Fuzzy ordered semigroup has been first considered by Kehayopulu and Tsingelis in [11]. In [12], the notions of pure fuzzy ideals of semigroups were introduced by Ahsan et al. In [13], Bashir et al. introduced and studied the notions of pure fuzzy ideal, purely fuzzy maximal and purely fuzzy prime ideals of ternary semigroups. The same authors gave some characterizations of pure fuzzy ideals and proved that each fuzzy ideal is weakly pure.

The purpose of this paper is to extend the results of Bashir et al. to ordered ternary semigroups. Section 2 recalls some certain definitions and results used throughout this paper. In Section 3, we introduce the concepts of pure fuzzy ideal in ordered ternary semigroups. We prove that an ideal $A$ of an ordered ternary semigroup $T$ is right pure ideal of $T$ if and only if the characteristic function of $A$ of $T$ is pure fuzzy ideal of $T$. And, the fuzzy ideal of $T$ is pure fuzzy ideal of $T$. In Sections 4 and 5, we characterize right weakly regular ordered ternary semigroups and weakly pure by fuzzy ideals. Finally, we end this paper with conclusion in Section 6.

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2. **Preliminaries.** A ternary semigroup is an algebraic structure $(T, [\ ])$ such that $T$ is a non-empty set and $[\ ] : T \times T \times T \to T$ a ternary operation satisfying associative law, that is,

$$[[a, b, c], d, e] = [a, [b, c, d], e] = [a, b, [c, d, e]]$$

for all $a, b, c, d, e \in T$. Moreover, for simplicity, we write $abc$ instead of $[a, b, c]$.

Any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup.

**Example 2.1.** *Let $T = \{\ldots, -2i, -i, 0, i, 2i, \ldots\}$. Then $T$ is a ternary semigroup under the multiplication over complex number.*

Observe that this ternary semigroup does not reduce to an ordinary semigroup under the same multiplication.

An *ordered ternary semigroup* is an algebraic structure $(T, [\ ], \leq)$ such that $(T, [\ ])$ is a ternary semigroup and $(T, \leq)$ is a partially ordered set satisfying the following condition:

$$a \leq b \Rightarrow acd \leq bcd,\ cad \leq cbd,\ cda \leq cdb$$

for all $a, b, c, d \in T$.

**Example 2.2.** *Let $\mathbb{Z}^-$ be an ordered ternary semigroup with respect to triple multiplication and partial ordering relation $\leq$.*

Let $T$ be an ordered ternary semigroup. An element $0$ of $T$ is called a *zero element* of $T$ if $000 = 0ab = a0b = ab0 = 0$ and $0 \leq a$ for all $a, b \in T$. An element $e$ of $T$ is called an *identity* (or *unital element*) if $eex = exe = xee = x$ for all $x \in T$.

Let $T$ be an ordered ternary semigroup. For a subset $A$ of $T$, let

$$(A] = \{t \in T \mid t \leq a \text{ for some } a \in A\}.$$

For subsets $A, B, C \subseteq T$, we have

(1) $A \subseteq (A]$,
(2) $A \subseteq B$ implies $(A] \subseteq (B]$,
(3) $((A]] = (A]$,
(4) $(A](B](C] \subseteq (ABC]$.

Let $T$ be an ordered ternary semigroup. A non-empty subset $A$ is called a *left* (respectively, *right, lateral*) *ideal* of $T$ if

(1) $TTA \subseteq A$ (respectively, $ATT$, $TAT$),
(2) if $a \in A$ and $t \in T$ such that $t \leq a$, then $t \in A$.

The second condition means that $A = (A]$. If $A$ is a left, a right and a lateral ideal of $T$, then $A$ is an *ideal* of $T$. An element $a$ of $T$ is said to be *regular* if there exists $x \in T$ such that $a \leq axa$, and $T$ is called *regular* if every element of $T$ is regular.

Let $T$ be an ordered ternary semigroup. An ideal $A$ of $T$ is called a *left* (*right*) *pure ideal* [6] if for each $a \in A$ there exist $x, y \in A$ such that $a \leq xya$ ($a \leq axy$).

Let $T$ be an ordered ternary semigroup. A fuzzy subset $f$ of $T$, i.e., the mapping from $T$ to the interval $[0, 1]$. A fuzzy subset $f$ of $T$ is called a *fuzzy ternary subsemigroup* of $T$ if (1) $x \leq y$ implies $f(x) \geq f(y)$ and (2) $f(xyz) \geq \min\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.

Let $T$ be an ordered ternary semigroup. A fuzzy subset $f$ of $T$ is called a *fuzzy left* (respectively, *right, lateral*) *ideal* of $T$ if

(1) $f(xyz) \geq f(z)$ (respectively, $f(xyz) \geq f(x)$, $f(xyz) \geq f(y)$) for all $x, y, z \in T$,
(2) $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \in T$.

Let $T$ be an ordered ternary semigroup. A fuzzy subset $f$ of $T$ is called a *fuzzy ideal* if

(1) $f(xyz) \geq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$,
(2) $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \in T$.

Let $f$ and $g$ be two fuzzy subsets of an ordered ternary semigroup $T$, and we define the relation $\preceq$ between $f$ and $g$, the union and the intersection of $f$ and $g$, respectively, as

$$f \preceq g \text{ if } f \leq g,$$
$$(f \vee g)(x) = \max\{f(x), g(x)\},$$
$$(f \wedge g)(x) = \min\{f(x), g(x)\},$$

for all $x \in T$. For $a \in T$, define

$$\mathcal{A}_a = \{(x, y, z) \in T \times T \times T \mid a \leq xyz\}.$$

For the three fuzzy subsets $f$, $g$ and $h$ of an ordered ternary semigroup $T$, we define the product as follows:

$$(f \circ g \circ h)(a) = \begin{cases} \sup\limits_{(x,y,z) \in \mathcal{A}_a} \{\min\{f(x), g(y), h(z)\}\} & \text{if } \mathcal{A}_a \neq \emptyset, \\ 0 & \text{if } \mathcal{A}_a = \emptyset. \end{cases}$$

We denote by $f_A$ the characteristic mapping function of subset $A$ of an ordered ternary semigroup $T$, that is the mapping of $T$ into $[0, 1]$ defined by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

3. **Pure Fuzzy Ideals in Ordered Ternary Semigroups.** In [13], Bashir et al. introduced pure fuzzy ideals of ternary semigroups. In this section, we define pure fuzzy ideals in ordered ternary semigroups.

**Definition 3.1.** *Let $T$ be an ordered ternary semigroup. A fuzzy ideal $\delta$ of $T$ is called a pure fuzzy ideal of $T$ if $\xi \wedge \delta = \xi \circ \delta \circ \delta$ for all right fuzzy ideals $\xi$ of $T$.*

**Lemma 3.1.** [6] *Let $T$ be an ordered ternary semigroup and $A$ an ideal of $T$. Then $A$ is a right pure ideal of $T$ if and only if $B \cap A = (BAA]$ for all right ideals $B$ of $T$.*

**Lemma 3.2.** [14] *Let $T$ be an ordered ternary semigroup and $\emptyset \neq A \subseteq T$. Then $A$ is a left ideal (respectively, right ideal, lateral ideal, ideal) of $T$ if and only if the characteristic function $f_A$ is a fuzzy left ideal (respectively, fuzzy right ideal, fuzzy lateral ideal, fuzzy ideal) of $T$.*

**Lemma 3.3.** [15] *Let $T$ be an ordered ternary semigroup, $n$ a natural number, $n \geq 2$ and $\{A_1, A_2, \ldots, A_n\}$ a set of non-empty subset of $T$. Then we have*

$$f_{A_1} \circ f_{A_2} \circ \cdots \circ f_{A_n} = f_{(A_1 A_2 \ldots A_n]}.$$

**Theorem 3.1.** *Let $T$ be an ordered ternary semigroup. Then an ideal $A$ of $T$ is right pure ideal of $T$ if and only if the characteristic function of $A$, denoted by $f_A$ of $T$ is pure fuzzy ideal of $T$.*

**Proof:** Assume that $A$ is a right pure ideal of $T$. Let $f_A$ be the characteristic function of $A$. To show that $f_A$ is a right pure ideal of $T$. Let $\xi$ be a fuzzy right ideal of $T$ and let $a \in T$. By Lemma 3.2, we have $f_A$ as a fuzzy ideal of $T$. Consider

$$\begin{aligned} (\xi \circ f_A \circ f_A)(a) &= \sup_{(x,y,z) \in \mathcal{A}_a} \{\min\{\xi(x), f_A(y), f_A(z)\}\} \\ &\leq \sup_{(x,y,z) \in \mathcal{A}_a} \{\min\{\xi(xyz), f_A(xyz), f_A(xyz)\}\} \\ &\leq \min\{\xi(a), f_A(a), f_A(a)\} \\ &= \xi(a) \wedge f_A(a) \wedge f_A(a) \\ &= (\xi \wedge f_A \wedge f_A)(a) \\ &= (\xi \wedge f_A)(a). \end{aligned}$$

This shows that $\xi \circ f_A \circ f_A \leq \xi \wedge f_A$. On the other hand, if $a \notin A$, then $(\xi \wedge f_A)(a) = 0$, whence $0 \leq (\xi \circ f_A \circ f_A)(a)$. We consider the case $a \in A$. Since $A$ is a right pure ideal, we get $a \leq auv$ for some $u, v \in A$. Then $\mathcal{A}_a \neq \emptyset$; moreover, $f_A(a) = 1 = f_A(u) = f_A(v)$. We have

$$\begin{aligned}(\xi \wedge f_A)(a) &= (\xi \wedge f_A \wedge f_A)(a) \\ &= \xi(a) \wedge f_A(a) \wedge f_A(a) \\ &= \xi(a) \wedge f_A(u) \wedge f_A(v) \\ &\leq \sup_{(a,u,v)\in\mathcal{A}_a} \{\min\{\xi(a), f_A(u), f_A(v)\}\} \\ &\leq \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(x), f_A(y), f_A(z)\}\} \\ &= (\xi \circ f_A \circ f_A)(a).\end{aligned}$$

Then $\xi \circ f_A \circ f_A \geq \xi \wedge f_A$. Hence, $\xi \circ f_A \circ f_A = \xi \wedge f_A$.

Conversely, assume that $f_A$ is a pure fuzzy ideal of $T$. To show that $A$ is a right pure ideal of $T$. Now, let $B$ be a right ideal of $T$. Then by Lemma 3.2, the characteristic function $f_B$ of $B$ is a fuzzy right ideal of $T$. By hypothesis, $f_B \wedge f_A = f_B \circ f_A \circ f_A$. By Lemma 3.3,

$$f_B \wedge f_A = f_B \circ f_A \circ f_A = f_{(BAA]}.$$

Accordingly, $B \cap A = (BAA]$. From Lemma 3.1, we have $A$ as a right pure ideal of $T$. □

**Theorem 3.2.** *Let $T$ be an ordered ternary semigroup with identity $e$. Then the fuzzy ideals $\phi$ and $f_T$ of $T$, defined respectively as*

$$\phi(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

*and $f_T(x) = 1$ for all $x \in T$, are pure fuzzy ideals of $T$.*

**Proof:** Let $\xi$ be a fuzzy right ideal of $T$. Consider if $a \neq 0$, then

$$(\xi \wedge \phi)(a) = \xi(a) \wedge \phi(a) = \xi(a) \wedge 0 = 0 \leq (\xi \circ \phi \circ \phi)(a).$$

If $a = 0$, then $\mathcal{A}_a \neq \emptyset$. We have

$$(\xi \wedge \phi)(0) = \xi(0) \wedge \phi(0) \leq \sup_{(x,y,z)\in\mathcal{A}_0} \{\min\{\xi(x), \phi(y), \phi(z)\}\} = (\xi \circ \phi \circ \phi)(0).$$

We deduce that $\xi \wedge \phi \leq \beta \circ \phi \circ \phi$. On the other hand, let $a \in T$. Observe that

$$\begin{aligned}(\xi \circ \phi \circ \phi)(a) &= \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(x), \phi(y), \phi(z)\}\} \\ &\leq \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(xyz), \phi(xyz), \phi(xyz)\}\} \\ &\leq \min\{\xi(a), \phi(a)\} \\ &= (\xi \wedge \phi)(a).\end{aligned}$$

It follows that $\xi \circ \phi \circ \phi \leq \xi \wedge \phi$, so $\xi \wedge \phi = \beta \circ \phi \circ \phi$. Thus, $\phi$ is a pure fuzzy ideal of $T$.

Now, we need show that $f_T$ is a pure fuzzy ideal of $T$. Observe that

$$\begin{aligned}(\xi \circ f_T \circ f_T)(a) &= \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(x), f_T(y), f_T(z)\}\} \\ &\leq \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(xyz), f_T(xyz), f_T(xyz)\}\} \\ &\leq \min\{\xi(a), f_T(a)\} \\ &= (\xi \wedge f_T)(a).\end{aligned}$$

This proves that $\xi \circ f_T \circ f_T \geq \xi \wedge f_T$. Besides,

$$\begin{aligned}(\xi \wedge f_T)(a) &= \min\{\xi(a), f_T(a)\} \\ &= \min\{\xi(a), f_T(a), f_T(a)\}\end{aligned}$$

$$\begin{aligned} &= \min\{\xi(a), f_T(e), f_T(e)\} \\ &\le \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(x), f_T(y), f_T(z)\}\} \\ &= (\xi \circ f_T \circ f_T)(a). \end{aligned}$$

Therefore, $\xi \circ f_T \circ f_T = \xi \wedge f_T$. □

**Theorem 3.3.** *If $\{\delta_i : i \in \Lambda\}$ is a family of pure fuzzy ideals of an ordered ternary semigroup $T$, then $\bigvee_{i\in\Lambda} \delta_i$ is pure fuzzy ideal of $T$.*

**Proof:** Let $\{\delta_i : i \in \Lambda\}$ be a family of pure fuzzy ideals of an ordered ternary semigroup $T$. Assume that $\xi$ is a fuzzy right ideal of $T$. Now let $a \in T$. We will show that

$$\left(\xi \wedge \left(\bigvee_{i\in\Lambda} \delta_i\right)\right)(a) = \left(\xi \circ \left(\bigvee_{i\in\Lambda} \delta_i\right) \circ \left(\bigvee_{i\in\Lambda} \delta_i\right)\right)(a).$$

Consider

$$\begin{aligned} &\left(\xi \circ \left(\bigvee_{i\in\Lambda} \delta_i\right) \circ \left(\bigvee_{i\in\Lambda} \delta_i\right)\right)(a) \\ &= \sup_{(x,y,z)\in\mathcal{A}_a} \left\{\min\left\{\xi(x), \left(\bigvee_{i\in\Lambda} \delta_i\right)(y), \left(\bigvee_{i\in\Lambda} \delta_i\right)(z)\right\}\right\} \\ &\le \sup_{(x,y,z)\in\mathcal{A}_a} \left\{\min\left\{\xi(xyz), \left(\bigvee_{i\in\Lambda} \delta_i\right)(xyz), \left(\bigvee_{i\in\Lambda} \delta_i\right)(xyz)\right\}\right\} \\ &\le \sup_{(x,y,z)\in\mathcal{A}_a} \left\{\min\left\{\xi(a), \left(\bigvee_{i\in\Lambda} \delta_i\right)(a), \left(\bigvee_{i\in\Lambda} \delta_i\right)(a)\right\}\right\} \\ &= \min\left\{\xi(a), \left(\bigvee_{i\in\Lambda} \delta_i\right)(a)\right\} \\ &= \left(\xi \wedge \left(\bigvee_{i\in\Lambda} \delta_i\right)\right)(a). \end{aligned}$$

This proves that $\xi \circ \left(\bigvee_{i\in\Lambda} \delta_i\right) \circ \left(\bigvee_{i\in\Lambda} \delta_i\right) \le \xi \wedge \left(\bigvee_{i\in\Lambda} \delta_i\right)$. Since $\delta_i$ are pure fuzzy ideals of $T$,

$$\begin{aligned} \bigvee_{i\in\Lambda}(\xi \circ \delta_i \circ \delta_i)(a) &= \bigvee_{i\in\Lambda}[(\xi \wedge \delta_i)(a)] \\ &= \bigvee_{i\in\Lambda}[\xi(a) \wedge \delta_i(a)] \\ &= \xi(a) \wedge \left(\bigvee_{i\in\Lambda} \delta_i(a)\right) \\ &= \left(\xi \wedge \left(\bigvee_{i\in\Lambda} \delta_i\right)\right)(a). \end{aligned}$$

Moreover,

$$(\xi \circ \delta_i \circ \delta_i)(a) = \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(x), \delta_i(y), \delta_i(z)\}\}$$

$$\leq \sup_{(x,y,z)\in\mathcal{A}_a} \left\{ \min \left\{ \xi(x), \left( \bigvee_{i\in\Lambda} \delta_i \right)(y), \left( \bigvee_{i\in\Lambda} \delta_i \right)(z) \right\} \right\}$$
$$= \left( \xi \circ \left( \bigvee_{i\in\Lambda} \delta_i \right) \circ \left( \bigvee_{i\in\Lambda} \delta_i \right) \right)(a).$$

Consequently, $\left( \xi \wedge \left( \bigvee_{i\in\Lambda} \delta_i \right) \right)(a) \leq \left( \xi \circ \left( \bigvee_{i\in\Lambda} \delta_i \right) \circ \left( \bigvee_{i\in\Lambda} \delta_i \right) \right)(a)$. The proof is completed. □

4. **Weakly Regular Ordered Ternary Semigroups.** In this section, we characterize right weakly regular ordered ternary semigroups by terms of fuzzy ideals.

**Definition 4.1.** [6] *An ordered ternary semigroup $T$ is called* right (left) weakly regular*, if for each $a \in T$ we obtain $a \in ((aTT)(aTT)(aTT)]$ $(a \in ((TTa)(TTa)(TTa)])$.*

For every regular ordered ternary semigroup induced by ordered semigroup it is right (left) weakly regular but the converse is not true.

As a consequence of the transfer principle for fuzzy sets, we have the following theorem.

Let $T$ be an ordered ternary semigroup. A fuzzy subset $f$ of $T$ is called an *idempotent* of $T$ if, $f \circ f \circ f = f^3 = f$.

**Theorem 4.1.** *Let $T$ be an ordered ternary semigroup. Then the following conditions are equivalent.*

(1) *$T$ is right weakly regular.*
(2) *Every fuzzy right ideal of $T$ is idempotent.*
(3) *Every fuzzy ideal of $T$ is a pure fuzzy ideal.*

**Proof:** (1)⇒(2) Assume that $T$ is right weakly regular. Let $\delta$ be a fuzzy right ideal of $T$. To show that $\delta$ is an idempotent. Now, let $a \in T$. Since $T$ is right weakly regular, $a \in ((aTT)(aTT)(aTT)]$. Then $a \leq (au_1u_2)(av_1v_2)(aw_1w_2)$ for some $u_1, u_2, v_1, v_2, w_1, w_2 \in T$; hence, $\mathcal{A}_a \neq \emptyset$. Then

$$\begin{aligned}(\delta \circ \delta \circ \delta)(a) &= \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\delta(x), \delta(y), \delta(z)\}\} \\ &\geq \min\{\delta(au_1u_2), \delta(av_1v_2), \delta(aw_1w_2)\} \\ &\geq \min\{\delta(a), \delta(a), \delta(a)\} \\ &= \delta(a).\end{aligned}$$

This shows that $\delta^3 \geq \delta$. Again, we have

$$\begin{aligned}(\delta \circ \delta \circ \delta)(a) &= \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\delta(x), \delta(y), \delta(z)\}\} \\ &\leq \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\delta(xyz), \delta(y), \delta(z)\}\} \\ &\leq \min\{\delta(a), \delta(a), \delta(a)\} \\ &= \delta(a).\end{aligned}$$

This implies that $\delta^3 \leq \delta$, whence $\delta^3 = \delta$.

(2)⇒(1) Assume that every fuzzy right ideal of $T$ is idempotent. Let $a \in T$ and put $R = (a \cup aTT]$. Then $R$ is a right ideal of $T$ containing $a$. Now, let $f_R$ be the characteristic function of $R$. By Lemma 3.2, $f_R$ is a fuzzy right ideal of $T$. We obtain that $f_R = f_R \circ f_R \circ f_R = f_{(RRR]}$ by assumption. Moreover, by Lemma 3.3, $R = (RRR]$. Consider

$$\begin{aligned}\left(R^3\right] &= ((a \cup aTT](a \cup aTT](a \cup aTT]] \\ &= \big(a^3 \cup aaTTa \cup aTTaa \cup aTTaTTa \cup aaaTT\end{aligned}$$

$$\cup aaTTaTT \cup aTTaaTT \cup aTTaTTaTT\big]$$
$$\subseteq \big((aTT)^3\big].$$

This shows that $a \in ((aTT)^3]$ because $a \in R = (R^3]$. Hence, $T$ is right weakly regular.

(1)⇒(3) Suppose that $T$ is right weakly regular. To show that every fuzzy ideal $\delta$ of $T$ is a pure fuzzy ideal. Let $\xi$ and $\delta$ be a fuzzy right ideal and a fuzzy ideal of $T$, respectively. Since $T$ is right weakly regular, there exist $u_1, u_2, v_1, v_2, w_1, w_2 \in T$ such that $a \le (au_1u_2)(av_1v_2)(aw_1w_2)$. That is $\mathcal{A}_a \neq \emptyset$. We have

$$\begin{aligned}(\xi \circ \delta \circ \delta)(a) &= \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(x), \delta(y), \delta(z)\}\} \\ &\ge \min\{\xi(au_1u_2), \delta(av_1v_2), \delta(aw_1w_2)\} \\ &\ge \min\{\xi(a), \delta(a), \delta(a)\} \\ &= \min\{\xi(a), \delta(a)\} \\ &= (\xi \wedge \delta)(a).\end{aligned}$$

This proves that $\xi \circ \delta \circ \delta \ge \xi \wedge \delta$. For the opposite direction, we have

$$\begin{aligned}(\delta \circ \delta \circ \delta)(a) &= \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(x), \delta(y), \delta(z)\}\} \\ &\le \sup_{(x,y,z)\in\mathcal{A}_a} \{\min\{\xi(xyz), \delta(xyz), \delta(xyz)\}\} \\ &\le \min\{\xi(a), \delta(a)\} \\ &= \xi \wedge \delta(a).\end{aligned}$$

That is $\xi \circ \delta \circ \delta \le \xi \wedge \delta$. Therefore, $\xi \circ \delta \circ \delta = \xi \wedge \delta$.

(3)⇒(1) Assume that every fuzzy ideal $\delta$ of $T$ is a pure fuzzy ideal. Now, let $a \in T$ and set $A = (a \cup aTT \cup TTa \cup TaT \cup TTaTT]$. Then $A$ is an ideal of $T$. Let $\delta_A$ be the characteristic function $\delta_A$ of $A$. Therefore by Lemma 3.2, $\delta_A$ is a fuzzy ideal of $T$. We have $\delta_A$ as a pure fuzzy ideal of $T$ by assumption. From Theorem 3.1, we obtain that $A$ is a right pure ideal of $T$. Observe that $a \in A$. Then there exists $x, y \in A$ such that $a \le axy$, because $A$ is a right pure ideal of $T$. It follows that $a \in (aAA]$. Observe that $(aAA] = (a(a \cup aTT \cup TTa \cup TaT \cup TTaTT](a \cup aTT \cup TTa \cup TaT \cup TTaTT]] = (aaa \cup aaaTT \cup aaTTa \cup aaTaT \cup aaTTaTT \cup aaTTa \cup aaTTaTT \cup aaTTTTa \cup aaTTTaT \cup aaTTTTaTT \cup aTTaa \cup aTTaaTT \cup aTTaTTa \cup aTTaTaT \cup aTTaTTaTT \cup aTaTa \cup aTaTaTT \cup aTaTTTa \cup aTaTTaT \cup aTaTTTaTT \cup aTTaTTa \cup aTTaTTaTT \cup aTTaTTTTa \cup aTTaTTTaT \cup aTTaTTTTaTT] \subseteq ((aTT)^3]$. Accordingly, $a \in ((aTT)^3]$, whence $T$ is right weakly regular. □

5. **Weakly Pure Fuzzy Ideals.** In this section, we introduce the concept weakly pure fuzzy ideals in ordered ternary semigroups.

**Definition 5.1.** *A fuzzy ideal $\delta$ of an ordered ternary semigroup $T$ is said to be left (right) weakly pure if $\delta \wedge \xi = \delta \circ \delta \circ \xi$ ($\xi \wedge \delta = \xi \circ \delta \circ \delta$) for all fuzzy ideals $\xi$ of $T$.*

**Theorem 5.1.** *Let $T$ be an ordered ternary semigroup. Then the following conditions are equivalent.*

(1) *Each every fuzzy ideal of $T$ is left weakly pure fuzzy.*
(2) *Each every fuzzy ideal of $T$ is idempotent.*
(3) *Each every fuzzy ideal of $T$ is right weakly pure fuzzy.*

**Proof:** (1)⇒(2) Assume that each every fuzzy ideal of $T$ is left weakly pure fuzzy. Let $\delta$ be a fuzzy ideal of $T$. Then by assumption, $\delta$ is left weakly pure fuzzy of $T$. Thus, $\delta \wedge \delta = \delta \circ \delta \circ \delta$; therefore, $\delta = \delta \wedge \delta = \delta \circ \delta \circ \delta$. This proves that $\delta$ is idempotent.

(2)⇒(1) Assume that every fuzzy ideal of $T$ is idempotent. Let $\delta$ and $\xi$ be fuzzy ideals of $T$. Then by hypothesis,

$$\xi \wedge \delta = (\xi \wedge \delta) \circ (\xi \wedge \delta) \circ (\xi \wedge \delta) \leq \delta \circ \delta \circ \xi.$$

Therefore, $\delta \wedge \xi \leq \delta \circ \delta \circ \xi$. On the other hand, we observe that $\delta \circ \delta \circ \xi \leq \delta \wedge \delta \wedge \xi = \xi \wedge \delta$. This proves that $\xi \wedge \delta = \delta \circ \delta \circ \xi$. Hence, $\delta$ is left weakly pure fuzzy.

(2)⇒(3) Assume that every fuzzy ideal is idempotent. Let $\xi$ and $\delta$ be fuzzy ideals of $T$. Then $\xi \wedge \delta$ is a fuzzy ideal of $T$. By assumption,

$$\xi \wedge \delta = (\xi \wedge \delta) \circ (\xi \wedge \delta) \circ (\xi \wedge \delta) \leq \xi \circ \delta \circ \delta.$$

On the other hand, we have $\xi \circ \delta \circ \delta \leq \xi \wedge \delta$ always. Thus, $\xi \wedge \delta = \xi \circ \delta \circ \delta$, whence $\delta$ is right weakly pure fuzzy.

(3)⇒(2) Assume that every fuzzy ideal of $T$ is right weakly pure fuzzy ideal of $T$. Let $\delta$ be a fuzzy ideal of $T$. Then $\delta$ is right weakly pure fuzzy ideal of $T$ by assumption. Hence, $\xi \wedge \delta = \xi \circ \delta \circ \delta$ for all fuzzy ideals $\xi$ of $T$. Indeed, $\delta = \delta \wedge \delta = \delta \circ \delta \circ \delta$. Hence, $\delta$ is idempotent of $T$. □

**Example 5.1.** *Let $T = \{\dots, -2, -1, 0\}$. Then $T$ is an ordered ternary semigroup with respect to triple multiplication and partial ordering relation $\leq$. Define*

$$\phi(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

*for all $x \in T$. By Theorem 3.2, we have $\phi$ as a pure fuzzy ideal of $T$. Hence, $\varphi$ is a weakly pure ideal and idempotent of $T$ by Theorem 5.1.*

## 6. Conclusion.

Fuzzy ideals are a generalization of ideals in ordered ternary semigroups. They can be used to characterize ordered ternary semigroups into classes. The present paper introduces the notion of purity for fuzzy ideals in ordered ternary semigroups. The characterizations of fuzzy ideals to satisfy purity are provided. Finally, we apply the purity of fuzzy ideals to characterizing ordered ternary semigroups.

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