ICIC Express Letters
Volume **16**, Number **2**, February **2022**

ICIC International ©2022 ISSN 1881-803X
pp. 127–135

ON ALMOST BI-IDEALS AND ALMOST QUASI-IDEALS OF ORDERED SEMIGROUPS AND THEIR FUZZIFICATIONS

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Received June 2021; accepted August 2021

Abstract. *In this paper, we introduce the concepts of almost bi-ideals, almost quasi-ideals, fuzzy almost bi-ideals, and fuzzy almost quasi-ideals in ordered semigroups. Moreover, we show that any nonempty subset A of an ordered semigroup S is an almost bi-ideal (resp. almost quasi-ideal) of S if and only if the characteristic function of A is a fuzzy almost bi-ideal (resp. fuzzy almost quasi-ideal) of S. Finally, we prove that any nonzero fuzzy subset f of an ordered semigroup S is a fuzzy almost bi-ideal (resp. fuzzy almost quasi-ideal) of S if and only if supp(f) is an almost bi-ideal (resp. almost quasi-ideal) of S.*

Keywords: Ordered semigroups, Almost bi-ideals, Almost quasi-ideals, Fuzzy almost bi-ideals, Fuzzy almost quasi-ideals

1. **Introduction.** Ideal theory plays an important role in studying in semigroups. The notion of almost ideals (or A-ideals) of semigroups was introduced by Grosek and Satko [1] in 1980. Afterwards, they discovered minimal almost ideals and smallest almost ideals of semigroups in [2,3], respectively. In 1981, Bogdanović [4] introduced the concept of almost bi-ideals in semigroups by using the notions of almost ideals and bi-ideals in semigroups. In 1965, Zadeh [5] introduced the concept of fuzzy subsets. Rosenfeld [6] applied the concept of Zadeh to defining fuzzy subgroups and fuzzy ideals in groups. In [7], Kuroki studied various kinds of fuzzy ideals in semigroups and characterized them. In 2002, Kehayopulu and Tsingelis introduced the notion of fuzzy ideals in ordered semigroups in [8]. In 2018, Wattanatripop et al. defined fuzzy almost bi-ideals of semigroups in [9]. Recently, Gaketem generalized results in [9] to study interval valued fuzzy almost bi-ideals of semigroups in [10]. Likewise, Wattanatripop et al. [11] examined almost quasi-ideals and provided the properties of almost quasi-ideals in semigroups. Ordered semigroups are one of generalizations of semigroups. In this paper, we focus on generalizing results of almost bi-ideals and almost quasi-ideals of semigroups to results of these almost ideals

DOI: 10.24507/icicel.16.02.127

in ordered semigroups. We define almost bi-ideals, almost quasi-ideals, fuzzy almost bi-ideals and fuzzy almost quasi-ideals in ordered semigroups. Moreover, we provide some properties of them and we study the relationship between them. Some results in [4, 9, 11] will become special cases of results of this paper.

2. **Preliminaries.** In this section, we give some definitions and results which will be used throughout this paper.

Definition 2.1. *A semigroup is a set S together with a binary operation $\cdot\colon S \times S \longrightarrow S$ that satisfies the associative property:*

$$\text{for all } x, y, z \in S,\ (xy)z = x(yz).$$

Definition 2.2. *Let S be a set with a binary operation $\cdot$ and a binary relation $\leq$. Then $(S, \cdot, \leq)$ is called an* ordered semigroup *if*
(1) $(S, \cdot)$ is a semigroup,
(2) $(S, \leq)$ is a partially ordered set, and
(3) for all $x, y, z \in S$, if $x \leq y$, then $xz \leq yz$ and $zx \leq zy$.

Definition 2.3. *An element a of an ordered semigroup S is called an* idempotent *if $a \leq a^2$.*

Let S be an ordered semigroup. For a nonempty subset A of S, we denote

$$(A] := \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Proposition 2.1. ([12]) *Let A and B be nonempty subsets of an ordered semigroup S.*
(1) $A \subseteq (A]$.
(2) If $A \subseteq B$, then $(A] \subseteq (B]$.
(3) $(A \cap B] \subseteq (A] \cap (B]$ and $(A \cup B] = (A] \cup (B]$.

For nonempty subsets A and B of an ordered semigroup S, we denote

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

Definition 2.4. *A nonempty subset A of an ordered semigroup S is called a* subsemigroup *of S if $AA \subseteq A$.*

Definition 2.5. *Let S be an ordered semigroup. A subsemigroup B of S is called a* bi-ideal *of S if*
(1) $BSB \subseteq B$ and
(2) if $x \in B$ and $s \in S$ such that $s \leq x$, then $s \in B$, that is, $(B] \subseteq B$.

Definition 2.6. *Let S be an ordered semigroup. A nonempty subset Q of S is called a* quasi-ideal *of S if*
(1) $(SQ] \cap (QS] \subseteq Q$ and
(2) if $x \in Q$ and $s \in S$ such that $s \leq x$, then $s \in Q$, that is, $(Q] \subseteq Q$.

Following the terminology given by Zadeh [5], let S be an ordered semigroup, we say that f is a fuzzy subset of S (or a fuzzy set in S) ([8]) if f is a mapping of S into the real closed interval $[0, 1]$. For any two fuzzy subsets f and g of a nonempty set S, the fuzzy subsets $f \vee g$ and $f \wedge g$ are defined as follows: for all $x \in S$,

$$(f \vee g)(x) = \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

Let f and g be two fuzzy subsets of an ordered semigroup S. In the set of all fuzzy subsets of S, we define the order relation "$\preceq$" as follows:

$$f \preceq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S.$$

For a fuzzy subset f of a nonempty set S, the *support* of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

The *characteristic mapping* of a subset A of a nonempty set S is a fuzzy subset of S defined by

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The definition of fuzzy points was given by Pu and Liu [13]. Let $s \in S$ and $\alpha \in (0, 1]$. A *fuzzy point* s_α of a set S is a fuzzy subset of S defined by

$$s_\alpha(x) = \begin{cases} \alpha & \text{if } x = s, \\ 0 & \text{otherwise.} \end{cases}$$

For an ordered semigroup S, the fuzzy subsets 1 and 0 of S are defined as follows:

$$1 : S \longrightarrow [0, 1] \mid x \longmapsto 1(x) := 1.$$
$$0 : S \longrightarrow [0, 1] \mid x \longmapsto 0(x) := 0.$$

Proposition 2.2. ([12]) *Let A and B be nonempty subsets of an ordered semigroup S. Then the following properties are true.*
(1) $C_A \wedge C_B \preceq C_{A\cap B}$.
(2) $C_A \vee C_B \preceq C_{A\cup B}$.
(3) If $A \subseteq B$, then $C_A \preceq C_B$.

Let $F(S)$ be the set of all fuzzy subsets of an ordered semigroup $(S, \cdot, \leq)$. For any $f, g \in F(S)$ and $x \in S$, we define the *product* of f and g by $f \circ g : S \longrightarrow [0, 1]$ such that

$$(f \circ g)(x) := \begin{cases} \sup\limits_{x \leq uv} \min\{f(u), g(v)\} & \text{if } x \leq uv \text{ where } u, v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.3. *Let f, g and h be fuzzy subsets of an ordered semigroup $(S, \cdot, \leq)$. Then the following properties hold.*
(1) If $f \preceq g$, then $f \circ h \preceq g \circ h$.
(2) If $f \preceq g$, then $f \wedge h \preceq g \wedge h$.
(3) If $f \preceq g$, then $f \vee h \preceq g \vee h$.
(4) If $f \preceq g$, then supp$(f) \preceq$ supp(g).

Definition 2.7. ([8]) *Let S be an ordered semigroup. A fuzzy set f of S is called a* fuzzy subsemigroup *of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.*

Definition 2.8. ([7]) *Let S be an ordered semigroup. A fuzzy subsemigroup f of S is called a* fuzzy bi-ideal *of S if for all $x, y, z \in S$,*
(1) if $x \leq y$, then $f(x) \geq f(y)$ and (2) $f(xyz) \geq \min\{f(x), f(z)\}$.

Definition 2.9. ([12]) *Let S be an ordered semigroup. A fuzzy subset f of S is called a* fuzzy quasi-ideal *of S if for all $x, y \in S$,*
(1) if $x \leq y$, then $f(x) \geq f(y)$ and (2) $(f \circ 1) \wedge (1 \circ f) \preceq f$.

From the above definition, the fuzzy bi-ideal is defined in terms of the fuzzy subset f itself while the fuzzy quasi-ideal in terms of the product $f \circ 1$ and $1 \circ f$. In [14], the fuzzy quasi-ideal f can be defined in a similar way using only the fuzzy subset f itself by the following theorem.

Theorem 2.1. ([14]) *Let S be an ordered semigroup. A fuzzy subset f of S is a fuzzy quasi-ideal of S if and only if the following conditions are satisfied.*
(1) If $x \leq y$, then $f(x) \geq f(y)$ for all $x, y \in S$.
(2) If $x \leq ab$ and $x \leq cd$, then $f(x) \geq \min\{f(a), f(d)\}$ for all $x, a, b, c, d \in S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. For a fuzzy subset f of S, we defined $(f] : S \longrightarrow [0,1]$ by

$$(f](x) = \sup_{x \leq y} f(y) \text{ for all } x \in S.$$

Proposition 2.4. *Let f, g and h be fuzzy subsets of an ordered semigroup S. Then the following statements hold.*
(1) $f \preceq (f]$.
(2) If $f \preceq g$, then $(f] \preceq (g]$.
(3) If $f \preceq g$, then $(f \circ h] \preceq (g \circ h]$ and $(h \circ f] \preceq (h \circ g]$.

Proposition 2.5. *Let f be a fuzzy subset of an ordered semigroup S. Then the following statements are equivalent.*
(1) If $x \leq y$, then $f(x) \geq f(y)$.
(2) $(f] = f$.

Note. For any $s \in S$ and $\alpha \in (0,1]$, we have the following conditions.
(1) $(s_\alpha \circ f] \wedge f \neq 0$ if and only if there exist $x, a \in S$ such that $x \leq sa$ and $f(x), f(a) \neq 0$.
(2) $(f \circ s_\alpha] \wedge f \neq 0$ if and only if there exist $x, a \in S$ such that $x \leq as$ and $f(x), f(a) \neq 0$.

3. **Almost Bi-Ideals and Almost Quasi-Ideals.** Throughout this paper, unless stated otherwise, S stands for an ordered semigroup. In this section, we define the notions of almost bi-ideals and almost quasi-ideals in ordered semigroups and some properties of them are provided.

Definition 3.1. *A nonempty subset B of S is called an* almost bi-ideal *of S if $(BxB] \cap B \neq \emptyset$ for all $x \in S$.*

Definition 3.2. *A nonempty subset Q of S is called an* almost quasi-ideal *of S if $(xQ] \cap (Qx] \cap Q \neq \emptyset$ for all $x \in S$.*

Example 3.1. *Consider the ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the order relation $\leq$ below.*

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leq := \{(a,a), (a,b), (b,b), (c,a), (c,b), (c,c), (d,a), (d,b), (d,d), (e,e)\}.$$

Then every nonempty subset of S is an almost bi-ideal of S except for $\{e\}$ and every nonempty subset of S except for $\{b\}, \{e\}$, and $\{b, e\}$ is an almost quasi-ideal of S.

Theorem 3.1. *The following properties hold in an ordered semigroup S.*
(1) If B is a bi-ideal of S, then B is an almost bi-ideal of S.
(2) If Q is a quasi-ideal of S and $xQ \cap Qx \neq \emptyset$ for all $x \in S$, then Q is an almost quasi-ideal of S.

Proof: (1) Let B be a bi-ideal of S and let $x \in S$. Then $BxB \neq \emptyset$ and $BxB \subseteq BSB \subseteq B$, so $(BxB] \subseteq (B] \subseteq B$. Thus, we have $\emptyset \neq BxB \subseteq (BxB] = (BxB] \cap B$. Hence $(BxB] \cap B \neq \emptyset$. Therefore, B is an almost bi-ideal of S. The proof of (2) is similar to the proof of (1). □

Example 3.2. *From Example 3.1, we can see that $\{a, b, c\}$ is an almost bi-ideal and an almost quasi-ideal of S but it is not a bi-ideal and a quasi-ideal of S because $(\{a, b, c\}] = \{a, b, c, d\} \not\subseteq \{a, b, c\}$.*

From the example above, in general, bi-ideals need not be an almost bi-ideal and quasi-ideals need not be an almost quasi-ideal.

Theorem 3.2. *Let A be a subset of S. If B is an almost bi-ideal (resp. almost quasi-ideal) of S such that $B \subseteq A$, then A is also an almost bi-ideal (resp. almost quasi-ideal) of S.*

Proof: Assume that B is an almost bi-ideal of S such that $B \subseteq A$. Let $x \in S$. Then $BxB \subseteq AxA$, so $(BxB] \subseteq (AxA]$. Thus, $(BxB] \cap B \subseteq (AxA] \cap A$. Since B is an almost bi-ideal of S, we have $(BxB] \cap B \neq \emptyset$. This implies that $(AxA] \cap A \neq \emptyset$. Hence A is an almost bi-ideal of S. The proof of the other is similar. □

Corollary 3.1. *The union of arbitrary set of almost bi-ideals (resp. almost quasi-ideals) of S is also an almost bi-ideal (resp. almost quasi-ideals) of S.*

Proof: This corollary follows from Theorem 3.2. □

In case of the intersection of two almost bi-ideals and of the intersection of two almost quasi-ideals, the result of the above corollary is not true. This can be seen by the following example.

Example 3.3. *From Example 3.1, we have $\{a, d, e\}$ and $\{b, c, e\}$ are almost bi-ideals and almost quasi-ideals of S. However, $\{a, d, e\} \cap \{b, c, e\} = \{e\}$ is not both an almost bi-ideal and an almost quasi-ideal of S.*

Lemma 3.1. *Let a be an element in S. Then the following statements hold.*

(1) $S - \{a\}$ is not an almost bi-ideal of S if and only if there exists an element $x \in S$ such that $((S - \{a\})x(S - \{a\})] = \{a\}$.

(2) $S - \{a\}$ is not an almost quasi-ideal of S if and only if there exists an element $x \in S$ such that $(x(S - \{a\})] \cap ((S - \{a\})x] \subseteq \{a\}$.

Proof: (1) Assume that $S - \{a\}$ is not an almost bi-ideal of S. Then there exists $x \in S$ such that $((S - \{a\})x(S - \{a\})] \cap (S - \{a\}) = \emptyset$, so we have $((S - \{a\})x(S - \{a\})] = \{a\}$. Conversely, assume that there exists an element $x \in S$ such that $((S - \{a\})x(S - \{a\})] = \{a\}$. Then $((S - \{a\})x(S - \{a\})] \cap (S - \{a\}) = \emptyset$. Thus, $S - \{a\}$ is not an almost bi-ideal of S. (2) can be proved in a similar manner. □

Theorem 3.3. *Let a be an element in S. Then the following statements hold.*

(1) If $S - \{a\}$ is not an almost bi-ideal of S, then either a or a^4 is an idempotent.

(2) If $S - \{a\}$ is not an almost quasi-ideal of S, then either a or a^2 is an idempotent.

Proof: (1) Assume that $S - \{a\}$ is not an almost bi-ideal of S. From Lemma 3.1(1), there exists $x \in S$ such that $((S - \{a\})x(S - \{a\})] = \{a\}$.

Case 1: $a = a^2$. Then $a = a^2 \leq a^2$, so a is an idempotent.

Case 2: $a \neq a^2$. That is, $a^2 \in S - \{a\}$, so $a^2xa^2 = a$.

Case 2.1: $x \leq a$. Then $a = a^2xa^2 \leq a^2aa^2 = a^5$, so $a^4 \leq a^8 = (a^4)^2$. Thus, a^4 is an idempotent.

Case 2.2: $x \not\leq a$. Then $x \in S - \{a\}$, so $x^3 = a$. Since $x \in S - \{a\}$ and $a^2 \in S - \{a\}$, we have $x^2a^2 = a$.

If $x^2a \leq a$, then $a = x^2a^2 = (x^2a)\,a \leq aa = a^2 \leq a^2$.

If $x^2a \not\leq a$, then $x^2a \in S - \{a\}$, so $x^2ax^3a = a$. Thus, we have

$$a = x^2a\left(x^3\right)a = x^2aaa = \left(x^2a^2\right)a = aa = a^2 \leq a^2.$$

Hence we can conclude that in Case 2.2, a is an idempotent. Therefore, a or a^4 is an idempotent.

(2) Assume that $S - \{a\}$ is not an almost quasi-ideal of S. By Lemma 3.1(2), there is an element $x \in S$ such that $(x(S - \{a\})] \cap ((S - \{a\})x] \subseteq \{a\}$.

Case 1: $a = a^2$. Then $a = a^2 \leq a^2$, so a is an idempotent.

Case 2: $a \neq a^2$. Then $a^2 \in S - \{a\}$, so $\big(xa^2\big] \cap \big(a^2x\big] \subseteq \{a\}$.

Case 2.1: $a \leq x$. Then $a^3 \leq xa^2$ and $a^3 \leq a^2x$, it follows that $a^3 \in \big(xa^2\big]$ and $a^3 \in \big(a^2x\big]$. So $a^3 \in \big(xa^2\big] \cap \big(a^2x\big]$. Thus, $a = a^3$, so $a^2 = a^4 \leq (a^2)^2$. In this case, a^2 is an idempotent.

Case 2.2: $a \not\leq x$. That is, $x \in S - \{a\}$. Then $x^2 \in \big(x^2\big] \subseteq \{a\}$, so $x^2 = a$. Thus, $a^2 = ax^2 = x^2a$.

If $a \leq xa$ and $a \leq ax$, then $a^2 \leq xa^2$ and $a^2 \leq a^2x$. This implies that $a^2 \in \big(xa^2\big]$ and $a^2 \in \big(a^2x\big]$, so $a^2 \in \big(xa^2\big] \cap \big(a^2x\big] \subseteq \{a\}$. Thus, $a = a^2 \leq a^2$.

If $a \leq xa$ and $a \not\leq ax$, then $a^2 \leq xa^2$ and $ax \in S - \{a\}$. Since $a^2 \in S - \{a\}$ and $ax \in S - \{a\}$, it follows that $\big(xa^2\big] \cap \big(ax^2\big] \subseteq \{a\}$. Since $a^2 \leq xa^2$ and $a^2 = ax^2$, we have $a^2 \in \big(xa^2\big] \cap \big(ax^2\big] \subseteq \{a\}$, so $a = a^2 \leq a^2$.

If $a \not\leq xa$ and $a \leq ax$, then $xa \in S - \{a\}$ and $a^2 \leq a^2x$. Since $xa \in S - \{a\}$ and $a^2 \in S - \{a\}$, it follows that $\big(x^2a\big] \cap \big(a^2x\big] \subseteq \{a\}$. Since $a^2 = x^2a$ and $a^2 \leq a^2x$, we have $a^2 \in \big(x^2a\big] \cap \big(a^2x\big] \subseteq \{a\}$, so $a = a^2 \leq a^2$.

If $a \not\leq xa$ and $a \not\leq ax$, then $xa, ax \in S - \{a\}$, so $\big(x^2a\big] \cap \big(ax^2\big] \subseteq \{a\}$. Since $a^2 = x^2a$ and $a^2 = ax^2$, we have $a^2 \in \big(x^2a\big] \cap \big(ax^2\big] \subseteq \{a\}$. Thus, $a = a^2 \leq a^2$.

In Case 2.2, we can see that $a \leq a^2$. Hence a is an idempotent.

Consequently, from all cases, we conclude that a or a^2 is an idempotent. □

4. **Fuzzy Almost Bi-Ideals and Fuzzy Almost Quasi-Ideals.** In this section, we introduce the concepts of fuzzy almost bi-ideals and fuzzy almost quasi-ideals in ordered semigroups. Moreover, some properties of fuzzy almost bi(quasi)-ideals are provided and the relationship between almost bi(quasi)-ideals and fuzzy almost bi(quasi)-ideals of ordered semigroups is studied.

Definition 4.1. *A nonzero fuzzy subset f of S is called a* fuzzy almost bi-ideal *of S if $(f \circ s_\alpha \circ f] \wedge f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.*

Definition 4.2. *A nonzero fuzzy subset f of S is called a* fuzzy almost quasi-ideal *of S if $(s_\alpha \circ f] \wedge (f \circ s_\alpha] \wedge f \neq 0$ for all $s \in S$ and $\alpha \in (0, 1]$.*

Note. Let $s \in S$ and $\alpha \in (0, 1]$. From the above definitions, we conclude that the following conditions hold.

(1) $(f \circ s_\alpha \circ f] \wedge f \neq 0$ if and only if there exist $x, a, b \in S$ such that $x \leq asb$ and $f(x), f(a), f(b) \neq 0$.
(2) $(s_\alpha \circ f] \wedge (f \circ s_\alpha] \wedge f \neq 0$ if and only if there exist $x, a, b \in S$ such that $x \leq sa$, $x \leq bs$ and $f(x), f(a), f(b) \neq 0$.

Example 4.1. *Consider the ordered semigroups $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the order relation $\leq$ below.*

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leq := \{(a, a), (a, b), (b, b), (c, a), (c, b), (c, c), (d, a), (d, b), (d, d), (e, e)\}.$$

Define functions $f\colon S \longrightarrow [0, 1]$ by $f(a) = 0$, $f(b) = 0.3$, $f(c) = 0$, $f(d) = 0.1$, $f(e) = 0.2$, and $g\colon S \longrightarrow [0, 1]$ by $g(a) = 0.1$, $g(b) = 0.3$, $g(c) = 0.1$, $g(d) = 0$, $g(e) = 0.3$.

Then we have f is a fuzzy almost bi-ideal, but it is not a fuzzy almost quasi-ideal of S, while g is both a fuzzy almost bi-ideal and a quasi-ideal of S.

Theorem 4.1. *Let f be a nonzero fuzzy subset of S. Then the following statements hold.*
(1) Every fuzzy bi-ideal of S is a fuzzy almost bi-ideal of S.
(2) If S is a commutative ordered semigroup, then every fuzzy quasi-ideal of S is a fuzzy almost quasi-ideal of S.

Proof: (1) Assume that f is a fuzzy bi-ideal of S. Let $s \in S$ and $\alpha \in (0,1]$. Since f is nonzero, then there exists an element $a \in S$ such that $f(a) \neq 0$. Let $x = asa$. Then we have

$$\begin{aligned}(f \circ s_\alpha \circ f](x) &= \sup_{x \le y}(f \circ s_\alpha \circ f)(y) \ge (f \circ s_\alpha \circ f)(x) = \sup_{x \le uv} \min\{(f \circ s_\alpha)(u), f(v)\} \\ &\ge \min\{(f \circ s_\alpha)(as), f(a)\} = \min\left\{\sup_{as \le uv} \min\{f(u), s_\alpha(v)\}, f(a)\right\} \\ &\ge \min\left\{\min\{f(a), s_\alpha(s)\}, f(a)\right\} = \min\{\alpha, f(a)\} \neq 0.\end{aligned}$$

Thus, $(f \circ s_\alpha \circ f](x) \neq 0$. Since f is a fuzzy bi-ideal of S, it follows that

$$f(x) = f(asa) \ge \min\{f(a), f(a)\} = f(a) \neq 0.$$

Hence $((f \circ s_\alpha \circ f] \wedge f)(x) = \min\{(f \circ s_\alpha \circ f](x), f(x)\} \neq 0$. This implies that $(f \circ s_\alpha \circ f] \wedge f \neq 0$. Therefore, f is a fuzzy almost bi-ideal of S.

(2) Assume that f is a fuzzy quasi-ideal of S. Let $s \in S$ and $\alpha \in (0,1]$. Since f is nonzero, then there is an element $a \in S$ such that $f(a) \neq 0$. Since S is commutative, it follows that $sa = as$. Let $x = sa = as$. Since $x \le sa$, then we have

$$\begin{aligned}(s_\alpha \circ f](x) &= \sup_{x \le y}(s_\alpha \circ f)(y) \ge (s_\alpha \circ f)(x) = \sup_{x \le uv} \min\{s_\alpha(u), f(v)\} \\ &\ge \min\{s_\alpha(s), f(a)\} = \min\{\alpha, f(a)\} \neq 0.\end{aligned}$$

Thus, $(s_\alpha \circ f](x) \neq 0$. Similarly, since $x \le as$, then we have $(f \circ s_\alpha](x) \neq 0$. Since f is a fuzzy quasi-ideal of S, then $x \le as$ and $x \le sa$, by Theorem 2.1, $f(x) \ge f(a) \neq 0$, so $f(x) \neq 0$. Hence $((s_\alpha \circ f] \wedge (f \circ s_\alpha] \wedge f)(x) \neq 0$. Therefore, f is a fuzzy almost quasi-ideal of S. □

Example 4.2. *From Example 4.1, we have g as a fuzzy almost bi-ideal and a fuzzy almost quasi-ideal of S. However, g is not a fuzzy bi-ideal and a fuzzy quasi-ideal of S because $a \le b$, but $f(a) = 0 \ngeq 0.3 = f(b)$.*

The above example shows that the converse of Theorem 4.1 is not true.

Theorem 4.2. *For every nonzero fuzzy subset f of S and g a fuzzy almost bi-ideal (resp. fuzzy almost quasi-ideal) of S such that $g \preceq f$, f is also a fuzzy almost bi-ideal (resp. fuzzy almost quasi-ideal) of S.*

Proof: Let f be a nonzero fuzzy subset of S and let $s \in S$ and $\alpha \in (0,1]$. Assume that g is a fuzzy almost bi-ideal of S such that $g \preceq f$. This implies that $(g \circ s_\alpha \circ g] \wedge g \neq 0$ and $(g \circ s_\alpha \circ g] \preceq (f \circ s_\alpha \circ f]$. Then we have $0 \neq (g \circ s_\alpha \circ g] \wedge g \preceq (f \circ s_\alpha \circ f] \wedge f$, so $(f \circ s_\alpha \circ f] \wedge f \neq 0$. Hence f is a fuzzy almost bi-ideal of S. The proof in other can be done in a similar way. □

Corollary 4.1. *The union of any two fuzzy almost bi-ideals (resp. fuzzy almost quasi-ideals) is also a fuzzy almost bi-ideal (resp. fuzzy almost quasi-ideal) of S.*

Proof: The proof follows from Theorem 4.2. □

Theorem 4.3. *A nonempty subset A of S is an almost bi-ideal (resp. almost quasi-ideal) of S if and only if C_A is a fuzzy almost bi-ideal (resp. fuzzy almost quasi-ideal) of S.*

Proof: Assume that A is an almost bi-ideal of S. Let $s \in S$ and $\alpha \in (0, 1]$. Then $(AsA] \cap A \neq \emptyset$. Thus, there exist $x \in A$ and $x \in (AsA]$, so $C_A(x) = 1 \neq 0$ and $x \leq asb$ for some $a, b \in A$. Since $a, b \in A$, then we have $C_A(a) = 1$ and $C_A(b) = 1$. This implies that $(C_A \circ s_\alpha \circ C_A](x) \geq \min\{C_A(a), s_\alpha(s), C_A(b)\} = \min\{1, \alpha, 1\} \neq 0$. Hence $((C_A \circ s_\alpha \circ C_A] \wedge C_A)(x) \neq 0$. Therefore, C_A is a fuzzy almost bi-ideal of S. Conversely, assume that C_A is a fuzzy almost bi-ideal of S and let $s \in S$. Choose $\alpha = 1$. Then $(C_A \circ s_1 \circ C_A] \wedge C_A \neq 0$, so there are $x, a, b \in S$ such that $x \leq asb$ and $C_A(x), C_A(a), C_A(b) \neq 0$. This implies that $x, a, b \in A$. We have $x \leq asb \in AsA$, so $x \in (AsA]$. Thus, $x \in (AsA] \cap A$. Hence $(AsA] \cap A \neq \emptyset$. Therefore, A is an almost bi-ideal of S. The proof in other can be done in a similar way. □

Theorem 4.4. *A nonzero fuzzy subset f of S is a fuzzy almost bi-ideal (resp. fuzzy almost quasi-ideal) of S if and only if supp(f) is an almost bi-ideal (resp. almost quasi-ideal) of S.*

Proof: Assume that f is a fuzzy almost bi-ideal of S. Let $s \in S$. Choose $\alpha = 1$. Then $(f \circ s_1 \circ f] \wedge f \neq 0$, so there are $x, a, b \in S$ such that $x \leq asb$ and $f(x), f(a), f(b) \neq 0$. Since $f(x), f(a), f(b) \neq 0$, it follows that $x, a, b \in supp(f)$. Thus, $x \leq asb \in (supp(f))s(supp(f))$, so we have $x \in ((supp(f))s(supp(f))]$. This implies that $x \in ((supp(f))s(supp(f))] \wedge supp(f)$. Therefore, $((supp(f))s(supp(f))] \wedge supp(f) \neq \emptyset$. Conversely, assume that $supp(f)$ is an almost bi-ideal of S. Let $s \in S$ and $\alpha \in (0, 1]$. Then $((supp(f))s(supp(f))] \wedge supp(f) \neq \emptyset$. That is, there exists $x \in S$ such that $x \in ((supp(f))s(supp(f))]$ and $x \in supp(f)$, so $x \leq asb$ and $f(x), f(a), f(b) \neq 0$. Thus, $(f \circ s_\alpha \circ f] \wedge f \neq 0$. Hence f is a fuzzy almost bi-ideal of S. The proof in other can be done in a similar way. □

5. **Conclusion.** The union of two almost bi-ideals (almost quasi-ideals) is also an almost bi-ideal (almost quasi-ideal) but the intersection of them need not be an almost bi-ideal (almost quasi-ideal) in ordered semigroups. In Theorems 4.3 and 4.4, we give some relationship between almost bi-ideals (almost quasi-ideals) and their fuzzifications. Moreover, the results in this paper generalized some results in [4, 9, 11].

In the future work, we can study other kinds of almost ideals and their fuzzifications in ordered semigroups or almost ideals and fuzzifications in other algebraic structures, for example, ordered Γ-semihypergroups were studied in [15].

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