ICIC Express Letters
Volume **16**, Number **5**, May **2022**

ICIC International ©2022 ISSN 1881-803X
pp. **469–476**

CHARACTERIZATIONS OF REGULARITIES ON ORDERED SEMIRINGS

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Received July 2021; accepted October 2021

Abstract. *We study many kinds of regularities on ordered semirings namely regular, left regular, right regular, intra-regular, completely regular, left weakly regular, right weakly regular, fully idempotent, left generalized regular, right generalized regular and generalized regular ordered semirings and investigate their connections. Some regularities can be characterized in terms of many kinds of ordered ideals of ordered semirings.*
Keywords: Semiring, Ordered semiring, Regular ordered semiring, Ordered ideal

1. **Introduction.** A semiring which is a common generalization of rings and distributive lattices, allowing the additive substructure to be only a semigroup instead of a group, appears in a natural manner in some applications to the theory of automata, formal languages, and other branches of applied mathematics (for example, see [1, 2, 3, 4]). To give an illustration, weighted automata on words were investigated by calculating that the weights form a semiring which is called a semiring-weighted automata [5]. Regularities are important properties to study on several algebraic structures such as semigroups, semirings and rings. They have been the target of considerable research about a century ago. In 1936, von Neumann [6] defined a regular ring as a ring $(S, +, \cdot)$ such that the semigroup $(S, \cdot)$ is regular and most researchers in area of rings theory usually call it von Neumann regularity. In 2011, Gan and Jiang [7] introduced the notion of an ordered semiring as a semiring together with a partially ordered relation connected by the compatibility property. Consequently, we are always to consider a semiring S as an ordered semiring together with the relation $\{(x, x) \in S \times S\}$. Later, in 2014, Mandal [8] gave the definition of a regular ordered semiring and characterized it using its fuzzy ideals. In our previous works [9, 10], we gave more characterizations of regular ordered semirings using ordered quasi-ideals, defined the notions of left regular, right regular and intra-regular ordered semirings and also characterized them in terms of ordered quasi-ideals.

In this work, we give more kinds of regularities on ordered semirings and study some of their algebraic properties. In Section 3, we recollect the notions of regular, left regular, right regular and intra-regular ordered semirings, present new types of regularities of ordered semirings, including, completely regular, left weakly regular, right weakly regular and generalized regular ordered semirings, investigate connections among all kinds of regularities mentioned above and give some of their examples. In Section 4, we characterize left regular, right regular, completely regular, left weakly regular and right weakly regular ordered semirings in terms of many kinds of their ordered ideals such as semiprime ordered quasi-ideals and semiprime ordered bi-ideals. Finally, in Section 5, we summarize the paper.

DOI: 10.24507/icicel.16.05.469

2. **Preliminaries.** A *semiring* is a system $(S,+,\cdot)$ consisting of a nonempty set S and two binary operations $+$ and $\cdot$ on S such that $(S,+)$ and $(S,\cdot)$ are semigroups connected by the distributive law. A semiring S is called *additively commutative* [11] if $a+b=b+a$ for all $a,b\in S$. An element 0 of a semiring S is called an *absorbing zero* [11] if $x+0=x=0+x$ and $x0=0=0x$ for all $x\in S$. An algebraic structure $(S,+,\cdot,\leq)$ is called an *ordered semiring* [7] if $(S,+,\cdot)$ is a semiring and $(S,\leq)$ is a partially ordered set satisfying the compatibility property, i.e., $a\leq b$ implies $a+c\leq b+c$, $c+a\leq c+b$, $ac\leq bc$ and $ca\leq cb$ for all $a,b,c\in S$.

In this work, we always assume that S is an additively commutative ordered semiring together with an absorbing zero 0.

For any nonempty subsets A and B of an ordered semiring S, we denote that $A+B=\{a+b\in S\mid a\in A \text{ and } b\in B\}$, $AB=\{ab\in S\mid a\in A \text{ and } b\in B\}$, $(A]=\{x\in S\mid x\leq a \text{ for some } a\in A\}$ and $\Sigma A=\big\{\sum_{i\in I}a_i\in S\mid a_i\in A \text{ and } I \text{ is a finite indexed set}\big\}$.

For any $a\in S$, we write Σa and $(a]$ instead of $\Sigma\{a\}$ and $(\{a\}]$, respectively. If $I=\emptyset$, we set $\sum_{i\in I}a_i=0$ for all $a_i\in S$. By basic properties of Σ and $(\ ]$, we refer to [12].

A nonempty subset A of S is called a *left ordered ideal* (resp. *right ordered ideal*) of S if $A+A\subseteq A$, $A=(A]$ and $SA\subseteq A$ (resp. $AS\subseteq A$). If A is both a left and a right ordered ideal of S, then A is called an *ordered ideal* [7] of S. A nonempty subset Q of S is called an *ordered quasi-ideal* [9] of S if $Q+Q\subseteq Q$, $Q=(Q]$ and $(\Sigma SQ]\cap(\Sigma QS]\subseteq Q$. A subsemiring B of S is called an *ordered bi-ideal* (resp. *ordered interior ideal*) [10] of S if $B=(B]$ and $BSB\subseteq B$ (resp. $SBS\subseteq B$).

For each element a of S, we denote the notations $L(a)$, $R(a)$ and $J(a)$ to be the left ordered ideal, right ordered ideal and ordered ideal of S generated by a, respectively. The following lemma is recalled from [9, 10].

Lemma 2.1. *Let $a\in S$. Then $L(a)=(\Sigma a+Sa]$, $R(a)=(\Sigma a+aS]$, and $J(a)=(\Sigma a+Sa+aS+\Sigma SaS]$.*

3. **Regularities of Ordered Semirings.** In this section, we recall and introduce some notions of regularities on ordered semirings and then we investigate their connections on ordered semirings.

Now, we recall some notions of regularities of ordered semiring as follows.

Definition 3.1. *An ordered semiring S is called* regular [8] *(resp.* left regular, right regular [9]*) if $a\in(aSa]$ (resp. $a\in(Sa^2]$, $a\in(a^2S]$) for all $a\in S$.*

Example 3.1. *Let $S=\{a,b,c,d,e\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:*

$+$	a	b	c	d	e
a	a	b	c	d	a
b	b	b	c	d	b
c	c	c	c	d	c
d	d	d	d	d	d
e	a	b	c	d	e

and

$\cdot$	a	b	c	d	e
a	b	b	b	b	e
b	b	b	b	b	e
c	c	c	c	c	e
d	c	c	c	d	e
e	e	e	e	e	e

Give a binary relation $\leq$ on S by $\leq:=\{(a,a),(b,b),(c,c),(d,d),(e,e),(a,c),(b,c)\}$. Then $(S,+,\cdot,\leq)$ is an ordered semiring. For each $x\in S\setminus\{a\}$, $x=x^2$ implies $x\in(Sx^2]$. Since $a\in(Sa^2]=\{a,b,c,e\}$, S is left regular. However, S is neither right regular nor regular because $a\notin(a^2S]=\{b,e\}=(aSa]$.

Example 3.2. *Consider $S=\{a,b,c,d,e\}$ with the operation $+$ and the relation $\leq$ defined in Example 3.1. Define a binary operation $\cdot$ on S by the following table:*

$\cdot$	a	b	c	d	e
a	b	b	c	c	e
b	b	b	c	c	e
c	b	b	c	c	e
d	b	b	c	d	e
e	e	e	e	e	e

Then $(S, +, \cdot, \leq)$ is an ordered semiring. For each $x \in S \setminus \{a\}$, $x = x^2$ implies $x \in (x^2S]$. Since $a \in (a^2S] = \{a, b, c, e\}$, S is right regular. However, S is neither left regular nor regular because $a \notin (Sa^2] = \{b, e\} = (aSa]$.

Example 3.3. *Let $S = \{a, b, c, d, e, f\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:*

$+$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	b	a
c	a	a	a	a	c	a
d	a	a	a	a	d	a
e	a	b	c	d	e	f
f	a	a	a	a	f	a

and

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	e	a
b	a	b	a	d	e	a
c	a	f	c	c	e	f
d	a	b	d	d	e	b
e	e	e	e	e	e	e
f	a	f	a	c	e	a

Define a binary relation $\leq$ on S by $\leq := \{(x, x) \mid x \in S\}$. Then $(S, +, \cdot, \leq)$ is an ordered semiring. For each $x \in S \setminus \{f\}$, $x = x^2$ implies $x \in (xSx]$. Since $f \in (fSf] = \{a, e, f\}$, S is regular. However, S is neither left regular nor right regular because $f \notin (Sf^2] = \{a, e\} = (f^2S]$.

In consequences of Examples 3.1, 3.2 and 3.3, we are able to say that the concepts of regular, left regular and right regular ordered semirings are independent.

We define the notion of a completely regular ordered semiring as a similar way of a completely regular ordered semigroup defined by Kehayopulu [13] as follows.

Definition 3.2. *An ordered semiring S is called* completely regular *if S is regular, left regular and right regular.*

Using Definitions 3.1 and 3.2, it is easy to obtain the following remark.

Remark 3.1. *An ordered semiring S is completely regular if and only if $a \in (a^2Sa^2]$ for all $a \in S$.*

We define the notion of an intra-regular ordered semiring as a generalization of an intra-regular semiring defined by Ahsan et al. [14] as follows.

Definition 3.3. *An ordered semiring S is called intra-regular if $a \in (\Sigma Sa^2S]$ for any $a \in S$.*

The regular ordered semiring defined as Example 3.3 is not intra-regular because $f \notin (\Sigma Sf^2S] = \{a, e\}$. On the other hand, the ordered semiring defined as Example 3.1 is intra-regular but not regular because $a \in (\Sigma Sa^2S] = \{a, b, c, e\}$ and $a \notin (aSa] = \{b, e\}$. Consequently, the concepts of regular and intra-regular ordered semirings are independent.

Remark 3.2. *Every left regular ordered semiring is intra-regular.*

Proof: Assume that S is left regular. Let $a \in S$. Then $a \in (Sa^2] \subseteq (S(Sa^2]a] \subseteq (Sa^3] \subseteq (Sa^2S] \subseteq (\Sigma Sa^2S]$. Hence, S is intra-regular. □

As a duality of Remark 3.2, we similarly obtain that every right regular ordered semiring is intra-regular.

The ordered semiring defined as Example 3.2 (resp. Example 3.1) is intra-regular but not left regular (resp. right regular).

Definition 3.4. [15] *An ordered semiring S is left weakly regular (resp. right weakly regular) if $a \in (\Sigma SaSa]$ (resp. $a \in (\Sigma aSaS]$) for all $a \in S$.*

Example 3.4. *Let $S = \{a, b, c, d\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:*

$+$	a	b	c	d
a	a	a	a	a
b	a	b	c	b
c	a	c	c	c
d	a	b	c	d

and

$\cdot$	a	b	c	d
a	a	a	a	d
b	b	b	b	d
c	a	a	a	d
d	d	d	d	d

Define a binary relation $\leq$ on S by $\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (c, b)\}$. Then $(S, +, \cdot, \leq)$ is an ordered semiring. For each $x \in S \setminus \{c\}$, $x = x^2$ implies $x \in (\Sigma SxSx]$. Since $c \in (\Sigma ScSc] = S$, S is left weakly regular. However, S is not right weakly regular because $c \notin (\Sigma cScS] = \{a, d\}$.

Example 3.5. *Consider $S = \{a, b, c, d\}$ with the operation $+$ and the relation $\leq$ defined as Example 3.4. Define a binary operation $\cdot$ on S by the following table:*

$\cdot$	a	b	c	d
a	a	b	a	d
b	a	b	a	d
c	a	b	a	d
d	d	d	d	d

Then $(S, +, \cdot, \leq)$ is an ordered semiring. For each $x \in S \setminus \{c\}$, $x = x^2$ implies $x \in (\Sigma xSxS]$. Since $c \in (\Sigma cScS] = S$, S is right weakly regular. However, S is not left weakly regular because $c \notin (\Sigma ScSc] = \{a, d\}$.

In consequences of Examples 3.4 and 3.5, the concepts of left and right weakly regularities are independent.

Remark 3.3. *Every left (resp. right) regular ordered semiring is left (resp. right) weakly regular.*

Proof: Assume that S is left regular. Let $a \in S$. Then $a \in (Sa^2] \subseteq (S(Sa^2]a] \subseteq (Sa^3] \subseteq (SaSa] \subseteq (\Sigma SaSa]$. Hence, S is left weakly regular. □

Example 3.6. *Consider the ordered semiring S defined in Example 3.3. For each $x \in S \setminus \{f\}$, $x = x^2$ implies $x \in (\Sigma SxSx]$ and $x \in (\Sigma xSxS]$. Since $f \in (\Sigma SfSf] = \{a, b, e, f\}$ and $f \in (\Sigma fSfS] = \{a, c, e, f\}$, we get that S is both left and right weakly regular. However, S is neither left nor right regular because $f \notin \{a, e\} = (Sf^2] = (f^2S]$.*

Remark 3.4. *Every regular ordered semiring is both left and right weakly regular.*

Proof: Assume that S is regular. Let $a \in S$. Then $a \in (aSa] \subseteq (aS(aSa]] \subseteq (aSaSa] \subseteq (SaSa] \subseteq (\Sigma SaSa]$. Hence, S is left weakly regular. □

The ordered semiring defined in Example 3.1 (resp. Example 3.2) is left (resp. right) weakly regular but not regular.

Definition 3.5. [15] *A nonempty subset T of an ordered semiring S is said to be* idempotent *if $T = (\Sigma T^2]$. If every ordered ideal of S is idempotent, then S is called* fully idempotent.

Lemma 3.1. *An ordered semiring S is fully idempotent if and only if $a \in (\Sigma SaSaS]$ for all $a \in S$.*

Proof: Assume that S is fully idempotent and let $a \in S$. Then, we obtain that

$$J(a) = (\Sigma J(a)J(a)] = \left(\Sigma \left(\Sigma(J(a))^2\right] \left(\Sigma(J(a))^2\right]\right] \subseteq \left(\Sigma(J(a))^4\right]$$

$$= \left(\Sigma(J(a))^3 J(a)\right] = \left(\Sigma(J(a))^3 \left(\Sigma(J(a))^2\right]\right] \subseteq \left(\Sigma(J(a))^5\right] \subseteq J(a).$$

It follows that $a \in J(a) = (\Sigma(J(a))^5] \subseteq (\Sigma SaSaS]$.

Conversely, let T be an ordered ideal of S. Clearly, $(\Sigma T^2] \subseteq (\Sigma ST] \subseteq (\Sigma T] = T$. Let $a \in T$. It follows that $a \in (\Sigma SaSaS] \subseteq (\Sigma STSTS] \subseteq (\Sigma TT] = (\Sigma T^2]$, i.e., $T \subseteq (\Sigma T^2]$. Hence, $T = (\Sigma T^2]$. Therefore, S is fully idempotent. $\square$

Remark 3.5. *Every intra-regular ordered semiring is fully idempotent.*

Proof: Let $a \in S$. Then $a \in (\Sigma Sa^2S] \subseteq (\Sigma Sa(\Sigma Sa^2S]S] \subseteq (\Sigma SaSa^2S] \subseteq (\Sigma SaSaS]$. By Lemma 3.1, S is fully idempotent. $\square$

Example 3.7. *Consider the ordered semiring S defined in Example 3.3. We have that $x = x^2$ for all $x \in S \setminus \{f\}$ and so $x = x^5 \in (\Sigma SxSxS]$ for all $x \in S \setminus \{f\}$. In addition, $f \in (\Sigma SfSfS] = S$. By Lemma 3.1, S is fully idempotent. However, $f \notin (\Sigma Sf^2S] = \{a, e\}$ implies that S is not intra-regular.*

Remark 3.6. *Every left (right) weakly regular ordered semiring is fully idempotent.*

Proof: Assume that S is left weakly regular and let $a \in S$. Then $a \in (\Sigma SaSa] \subseteq (\Sigma SaS(\Sigma SaSa]] \subseteq (\Sigma SaSaS]$. By Lemma 3.1, S is fully idempotent. $\square$

The ordered semiring S defined in Example 3.2 (resp. Example 3.1) is also fully idempotent but not left (resp. right) weakly regular.

Definition 3.6. *An ordered semiring S is generalized regular (resp. left generalized regular, right generalized regular) if $a \in (\Sigma SaS]$ (resp. $a \in (Sa]$, $a \in (aS]$) for all $a \in S$.*

Remark 3.7. *Every fully idempotent ordered semiring is generalized regular.*

Proof: Assume that S is fully idempotent and let $a \in S$. By Lemma 3.1, $a \in (\Sigma SaSaS] \subseteq (\Sigma SaS]$. Hence, S is generalized regular. $\square$

Example 3.8. *Let $S = \{a, b, c, d\}$. Define a binary operation $+$ on S by $d+c = b = c+d$, $x+x = x$, $x+a = x = a+x$ and $x+b = b = b+x$ for all $x \in S$. Define a binary operation $\cdot$ on S by $bb = bc = bd = cb = cc = db = b$, $cd = dc = c$, $dd = d$ and $xa = a = ax$ for all $x \in S$. Define a binary relation $\leq$ on S by $\leq := \{(a, a), (b, b), (c, c), (d, d), (b, c)\}$. Then $(S, +, \cdot, \leq)$ is an ordered semiring. For each $x \in S \setminus \{c\}$, $x = x^2$ implies $x \in (\Sigma SxS]$. Since $c \in (\Sigma ScS] = \{a, b, c\}$, S is generalized regular. However, S is not fully idempotent because $c \notin (\Sigma ScScS] = \{a, b\}$.*

The proof of the following remark is routine.

Remark 3.8. *Every left (resp. right) weakly regular ordered semiring is left (resp. right) generalized regular.*

The ordered semiring S defined in Example 3.8 is both left and right generalized regular because $c \in (Sc] = (cS] = \{a, b, c\}$ and $x = x^2 \in (Sx] = (xS]$ for all $x \in S \setminus \{c\}$. However, S is neither left nor right weakly regular because $c \notin (\Sigma ScSc] = (\Sigma cScS] = \{a, b\}$.

4. **Some Characterizations of Regularities.** In this section, we characterize some types of regularities on ordered semirings in terms of many kinds of their ordered ideals.

Theorem 4.1. *An ordered semiring S is completely regular if and only if every ordered quasi-ideal of S is a completely regular subsemiring of S.*

Proof: Assume that S is completely regular. Let Q be an ordered quasi-ideal of S and let $a \in Q$. Using Remark 3.1 and the fact that every ordered quasi-ideal is an ordered bi-ideal [9], we obtain that $a \in (a^2Sa^2] \subseteq (a(a^2Sa^2]S(a^2Sa^2]a] \subseteq (a^3Sa^3] \subseteq (a^2QSQa^2] \subseteq (a^2Qa^2]$. Using Remark 3.1 again, we get that Q is completely regular. The converse is clear because S itself is an ordered quasi-ideal. $\square$

As consequences of Theorem 4.1 and the fact that ordered quasi-ideals and ordered bi-ideals coincide in a regular ordered semiring [9], we obtain the following corollary.

Corollary 4.1. *An ordered semiring S is completely regular if and only if every ordered bi-ideal of S is a completely regular subsemiring of S.*

Definition 4.1. *A nonempty subset T of S is called* semiprime *if for each $a \in S$, $a^2 \in T$ implies $a \in T$.*

The following theorem is a characterization of an ordered semiring which is both left and right regular by semiprime ordered quasi-ideals.

Theorem 4.2. *An ordered semiring S is both left and right regular if and only if every ordered quasi-ideal of S is semiprime.*

Proof: Assume that S is both left regular and right regular. Let Q be an ordered quasi-ideal of S and let $a \in S$ such that $a^2 \in Q$. Then $(a^2S] \cap (Sa^2]$ is an ordered quasi-ideal of S. By assumption, $a \in (a^2S] \cap (Sa^2] \subseteq (QS] \cap (SQ] \subseteq Q$ and so Q is semiprime.

Conversely, assume that every ordered quasi-ideal of S is semiprime and let $x \in S$. Since $x^4 \in x^2SS \subseteq (x^2S]$ and $x^4 \in SSx^2 \subseteq (Sx^2]$, we get $x^4 \in (x^2S] \cap (Sx^2]$. Since $(x^2S] \cap (Sx^2]$ is a semiprime ordered quasi-ideal of S, we get that $x^2 \in (x^2S] \cap (Sx^2]$ and $x \in (x^2S] \cap (Sx^2]$ as well. Therefore, S is both left regular and right regular. $\square$

The following example is an example of an ordered semiring in which all of its ordered quasi-ideals are semiprime but it is not completely regular.

Example 4.1. *Let $S = \{a, b, c, d, e, f, g\}$. Define a binary operation $+$ on S by $x + e = x = x + e$ for all $x \in S$ and $y + z = a$ for all $y, z \in S \setminus \{e\}$. Define a binary operation $\cdot$ on S by the following table:*

$\cdot$	a	b	c	d	e	f	g
a	a	a	a	a	e	a	a
b	a	b	b	d	e	b	d
c	a	b	b	d	e	b	d
d	a	b	b	d	e	b	d
e	e	e	e	e	e	e	e
f	a	f	f	g	e	f	g
g	a	f	f	g	e	f	g

Define a binary relation $\leq$ on S by

$$\leq := \{(x, x) \mid x \in S\} \cup \{(b, c), (b, d), (b, f), (b, g), (c, d), (c, f), (c, g), (d, g), (f, g)\}.$$

Then $(S, +, \cdot, \leq)$ is an ordered semiring. We have that $\{e\}$, $\{a, e\}$, $\{a, b, c, e\}$, $\{a, b, c, d, e\}$, $\{a, b, c, e, f\}$ and S are all ordered quasi-ideals of S. It is not difficult to see that all of these ordered quasi-ideals are semiprime. Then by Theorem 4.2, S is both left and right regular. However, S is not completely regular because S is not regular; in fact, $c \notin \{a, b, e\} = (cSc]$.

Theorem 4.3. *An ordered semiring S is completely regular if and only if every ordered bi-ideal of S is semiprime.*

Proof: Assume that S is completely regular. Then S is regular and so every ordered bi-ideal is an ordered quasi-ideal. Using Theorem 4.2, we have that every ordered bi-ideal of S is also semiprime.

Conversely, let $a \in S$. We claim that $(a^2Sa^2]$ is an ordered bi-ideal of S. It is easy to check that $(a^2Sa^2]$ is a subsemiring of S. We get that $(a^2Sa^2]S(a^2Sa^2] \subseteq (a^2S]S(Sa^2] \subseteq (a^2SS](Sa^2] \subseteq (a^2S](Sa^2] \subseteq ((a^2S)(Sa^2)] \subseteq (a^2Sa^2]$. Clearly, $((a^2Sa^2]] = (a^2Sa^2]$. Thus, $(a^2Sa^2]$ is an ordered bi-ideal of S. We have that $a^8 = a^2a^4a^2 \in (a^2Sa^2]$. By assumption, $(a^2Sa^2]$ is semiprime. It leads to $a^4 \in (a^2Sa^2]$, $a^2 \in (a^2Sa^2]$ and $a \in (a^2Sa^2]$. By Remark 3.1, S is completely regular. $\square$

Lemma 4.1. *An ordered semiring S is left weakly regular if $a \in (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]$ for any $a \in S$.*

Proof: Let $a \in S$. Assume that

$$a \in \left(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa\right] \quad (1)$$

$$\subseteq (Sa + \Sigma SaSa]. \quad (2)$$

Using the condition (2), we obtain that

$$a^2 = aa \subseteq (Sa + \Sigma SaSa](Sa + \Sigma SaSa] \subseteq (\Sigma SaSa]. \quad (3)$$

Using the condition (2) again, we obtain that

$$aSa \subseteq (Sa + \Sigma SaSa]S(Sa + \Sigma SaSa] \subseteq (\Sigma SaSa]. \quad (4)$$

Using the condition (3), we obtain that

$$Sa^2 \subseteq S(\Sigma SaSa] \subseteq (S\Sigma SaSa] \subseteq (\Sigma SaSa]. \quad (5)$$

Using the conditions (1), (3), (4) and (5), we obtain that

$$\begin{aligned} a &\in \left(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa\right] \\ &\subseteq (\Sigma(\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa] + \Sigma SaSa] \\ &\subseteq ((\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa]] \\ &\subseteq (\Sigma SaSa]. \end{aligned}$$

Hence, S is left weakly regular. □

Theorem 4.4. *The following statements are equivalent.*

(*i*) *S is left weakly regular.*
(*ii*) *$L \cap B \subseteq (\Sigma LSB]$ for every left ordered ideal L and ordered bi-ideal B of S.*
(*iii*) *$L = (\Sigma L^2]$ for every left ordered ideal L of S.*
(*iv*) *$I \cap L \subseteq (\Sigma IL]$ for every ordered interior ideal I and left ordered ideal L of S.*
(*v*) *$J \cap L \subseteq (\Sigma JL]$ for every ordered ideal J and left ordered ideal L of S.*

Proof: $(i) \Rightarrow (ii)$ Assume that S is left weakly regular and let L and B be a left ordered ideal and an ordered bi-ideal of S, respectively. If $x \in L \cap B$, then $x \in (\Sigma SxSx] \subseteq (\Sigma SLSB] \subseteq (\Sigma LSB]$.

$(ii) \Rightarrow (iii)$ It follows from the fact that every left ordered ideal is an ordered bi-ideal.

$(iii) \Rightarrow (i)$ Let $a \in S$. Using (iii) and Lemma 2.1, we obtain that

$$\begin{aligned} a \in L(a) = (\Sigma L(a)L(a)] &= (\Sigma(\Sigma a + Sa](\Sigma a + Sa]] \\ &\subseteq \left(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa\right]. \end{aligned}$$

By Lemma 4.1, we obtain that S is left weakly regular.

$(i) \Rightarrow (iv)$ Assume that S is left weakly regular and let I and L be an ordered interior ideal and a left ordered ideal of S, respectively. If $x \in I \cap L$, then $x \in (\Sigma SxSx] \subseteq (\Sigma SISL] \subseteq (\Sigma IL]$.

$(iv) \Rightarrow (v)$ It follows from the fact that every ordered ideal of an ordered semiring is an ordered interior ideal.

$(v) \Rightarrow (i)$ Let $a \in S$. Using (v), and Lemma 2.1, we obtain that

$$\begin{aligned} a \in J(a) \cap L(a) \subseteq (\Sigma J(a)L(a)] &= (\Sigma(\Sigma a + aS + Sa + \Sigma SaS](\Sigma a + Sa]] \\ &\subseteq \left(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa\right]. \end{aligned}$$

By Lemma 4.1, we obtain that S is left weakly regular. □

As a duality of Lemma 4.1 and Theorem 4.4, characterizations of a right weakly regular ordered semiring can be obtained analogously.

5. **Conclusion and Discussion.** In this work, we study eleven kinds of regularities on ordered semirings and their connections are investigated in Section 3. The concept of the completely regular ordered semiring is a special case of all kinds of regularities mentioned in this work. Moreover, it can be characterized in terms of semiprime ordered bi-ideals by the result that an ordered semiring is completely regular if and only if every ordered bi-ideal is semiprime. This result is also similarly obtained in an ordered semigroup [13]. However, there are many kinds of regularities on an ordered semiring such that the addition plays an important role such as ordered k-regularities [12, 16, 17, 18, 19]. The such kinds of regularities are special to study on ordered semirings and cannot be studied on semigroups because both addition and multiplication are indispensable.

Acknowledgment. This work has received scholarship under the Post Doctoral Training Program from Khon Kaen University, Thailand.

REFERENCES

[1] K. Glazek, *A Guide to Literature on Semirings and Their Applications in Mathematics and Information Sciences with Complete Bibliography*, Kluwer Academic Publishers, Dodrecht, 2002.
[2] J. S. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, Dodrecht, 1999.
[3] U. Hebisch and H. J. Weinert, *Semirings: Algebraic Theory and Applications in the Computer Science*, World Scientific, Singapore, 1998.
[4] W. Kuich and W. Salomma, *Semirings, Automata, Languages*, Springer Verlag, Berlin, 1986.
[5] M. Droste and B. Pibaljommee, Weighted nested word automata and logics over strong bimonoids, *International Journal of Foundations of Computer Science*, vol.25, no.5, pp.641-666, 2014.
[6] J. von Neumann, On regular rings, *Proc. Natl. Acad. Sci. USA*, vol.22, pp.707-713, 1936.
[7] A. P. Gan and Y. L. Jiang, On ordered ideals in ordered semirings, *J. Math. Res. Exposition*, vol.31, no.6, pp.989-996, 2011.
[8] D. Mandal, Fuzzy ideals and fuzzy interior ideals in ordered semirings, *Fuzzy Inf. Eng.*, vol.6, no.1, pp.101-114, 2014.
[9] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of regular ordered semirings by ordered quasi-ideals, *Int. J. Math. Math. Sci.*, 4272451, 2016.
[10] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of intra-regular ordered semirings by ordered quasi-ideals, *Proc. of the National and International Graduate Research Conference*, Khon Kaen University, Khon Kaen, Thailand, pp.102-109, 2016.
[11] M. R. Adhikari and A. Adhikari, *Basic Modern Algebra with Applications*, Springer Publication, New Delhi, 2014.
[12] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of ordered k-regular semirings by ordered quasi k-ideals, *Quasigroups Related Systems*, vol.25, no.1, pp.109-120, 2017.
[13] N. Kehayopulu, On completely regular ordered semigroups, *Sci. Math.*, vol.1, no.1, pp.27-32, 1998.
[14] J. Ahsan, N. Mordeson and M. Shabir, *Fuzzy Semirings with Applications to Automata Theory*, Springer Publication, Berlin, 2012.
[15] K. Siribute, P. Palakawong na Ayutthaya and J. Seanborisoot, Characterizations of regularities on ordered semirings by idempotency of ordered ideals, *Quasigroups Related Systems*, vol.29, no.1, pp.133-144, 2021.
[16] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of completely ordered k-regular semirings, *Songklanakarin J. Sci. Technol.*, vol.41, no.3, pp.501-505, 2019.
[17] P. Palakawong na Ayuthaya and B. Pibaljommee, Characterizations of ordered k-regularities on ordered semirings, *Quasigroups Related Systems*, vol.29, no.1, pp.107-121, 2021.
[18] S. Patchakhieo and B. Pibaljommee, Characterizations of ordered k-regular semirings by ordered k-ideals, *Asian-European J. Math.*, vol.10, 4272451, 2017.
[19] R. P. Sharma, M. Dadhwal, R. Sharma and S. Kar, On the primary decomposition of k-ideals and fuzzy k-ideals in semirings, *Fuzzy Inf. Eng.*, vol.13, pp.1-13, 2021.