ON ORDERED LA-Γ-SEMIGROUPS CONTAINING TWO-SIDED BASES

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Abstract. Two-sided base is the smallest set generated two-sided ideal under some condition. The aim of this paper is to introduce the concept of two-sided bases of an ordered LA-Γ-semigroup with left identity. We give a characterization when a non-empty subset of an ordered LA-Γ-semigroup with left identity is a two-sided base of an ordered Γ-semigroup with left identity. Finally, a characterization when the complement of the union of all two-sided bases of an ordered Γ-semigroup with left identity is maximal will be given.

Keywords: Ordered LA-Γ-semigroups, Γ-ideals, Two-sided bases, Maximal proper Γ-ideals

1. Introduction. Based on the notion of two-sided ideals of a semigroup generated by a non-empty set, the concept of two-sided bases of a semigroup has been introduced and studied by Fabrici [1]. Later, Changpas and Kummoon [2] studied and described the structure of a Γ-semigroup containing two-sided bases. The structure of a Γ-semigroup was introduced by Sen [3] as a generalization of ternary semigroup and semigroup and the structure of an LA-semigroup was introduced by Kazim and Naseeruddin [4] as a generalization of commutative semigroups. The structure of an LA-Γ-semigroups (Γ-AG-groupoid), where Γ is a non-empty set, was given by Shah and Rehman [5]. The concept of an ordered LA-Γ-semigroups was introduced by Khan et al. [6]. This algebraic structure is a generalization of LA-Γ-semigroups, also see [7, 8]. The purpose of this paper is to introduce the concept of two-sided bases of an ordered LA-Γ-semigroup, and extend results in [1] to ordered LA-Γ-semigroups. In Section 2, we recall some basic definitions and results of ordered LA-Γ-semigroups. In Section 3, we define two-sided bases of ordered LA-Γ-semigroups and give their basic results. Section 4 is the main part of this paper, and we show remarkable results of two-sided bases of ordered LA-Γ-semigroups. Finally, Section 5 concludes the paper.

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2. Ordered LA-Γ-Semigroups. We provide some definitions and results which will be used for this paper.

**Definition 2.1.** ([5]) Let $S$ and $\Gamma$ be non-empty sets, then $S$ is called an LA-Γ-semigroup if there exists a mapping $S \times \Gamma \times S \to S$ written as $(a, \gamma, b)$ and denoted by $a \gamma b$ such that $S$ satisfies the left invertive law $(a \gamma b) \beta c = (c \gamma b) \beta a$ for all $a, b, c \in S$ and $\gamma, \beta \in \Gamma$.

**Definition 2.2.** ([5]) An element $e$ of an LA-Γ-semigroup $S$ is called a left identity if $e \gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.

**Lemma 2.1.** ([5]) If $S$ is an LA-Γ-semigroup with left identity $e$, then $STS = S$ and $S = e\Gamma S = S\Gamma e$.

**Proposition 2.1.** ([9]) Let $S$ be an LA-Γ-semigroup.

1. Every LA-Γ-semigroup with left identity satisfies the equalities $a \gamma (b\beta c) = b\gamma (a\beta c)$ and $(a \gamma b) \beta (c\gamma d) = (d\gamma c) \beta (b\alpha a)$ for all $a, b, c, d \in S$ and $\gamma, \beta, \alpha \in \Gamma$.

2. An LA-Γ-semigroup $S$ is $\Gamma$-medial, i.e., $(a \gamma b) \beta (c \gamma d) = (a \gamma c) \beta (b \alpha d) = (a \gamma c) \beta (b \alpha d)$ for all $a, b, c, d \in S$ and $\gamma, \beta, \alpha \in \Gamma$.

**Definition 2.3.** ([6]) An ordered LA-Γ-semigroup $S$ (abbreviated as a po-LA-Γ-semigroup) is a structure $(S, \Gamma, \cdot, \leq)$ in which the following conditions hold.

1. $(S, \Gamma, \cdot)$ is an LA-Γ-semigroup.
2. $(S, \leq)$ is a poset (i.e., reflexive, anti-symmetric and transitive).
3. For all $a, b$ and $x \in S$, $a \leq b$ implies $a \alpha x \leq b \alpha x$ and $x \alpha a \leq x \alpha b$ for all $\alpha \in \Gamma$.

Throughout this paper, unless stated otherwise, $S$ stands for an ordered LA-Γ-semigroup. For a non-empty subsets $A, B$ of an ordered LA-Γ-semigroup $S$, we defined

$$A \Gamma B = \{a \gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\} \text{ and } (A) = \{t \in S \mid t \leq a, \text{ for some } a \in A\}.$$

In particular, we write $B \Gamma a$ instead for $B \Gamma \{a\}$, $a \Gamma B$ instead for $\{a\} \Gamma B$, $a \cup B \Gamma a \cup a \Gamma s \cup (S \Gamma a) \Gamma S$ instead for $\{a\} \cup B \Gamma a \cup a \Gamma s \cup (S \Gamma a) \Gamma S$ and $[a]$ instead for $(\{a\}]$.

**Definition 2.4.** ([7]) A non-empty subset $A$ of an ordered LA-Γ-semigroup $S$ is called an LA-Γ-subsemigroup of $S$ if $A \Gamma A \subseteq A$.

**Definition 2.5.** ([6]) A non-empty subset $A$ of an ordered LA-Γ-semigroup $S$ is called a left (resp. right) $\Gamma$-ideal of $S$ if (i) $S \Gamma A \subseteq A$ ($A \Gamma S \subseteq A$) and (ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$. A non-empty subset $A$ of an ordered LA-Γ-semigroup $S$ is called a $\Gamma$-ideal of $S$ if is both a left and right $\Gamma$-ideal of $S$.

**Definition 2.6.** A proper $\Gamma$-ideal $A$ of an ordered LA-Γ-semigroup $SA \neq S$ is said to be maximal if for any $\Gamma$-ideal $B$ of $S$, $A \subseteq B \subseteq S$ implies $A = B$ or $B = S$.

**Lemma 2.2.** ([6]) Let $S$ be an ordered LA-Γ-semigroup, and then the following statements are true.

1. $A \subseteq [A]$, for all $A \subseteq S$.
2. If $A \subseteq B \subseteq S$ then $[A] \subseteq [B]$.
3. $[A] \Gamma [B] \subseteq [A \Gamma B]$, for all subsets $A, B$ of $S$.
4. $[A] = ([A])$, for all $A \subseteq S$.
5. For every left (resp. right) $\Gamma$-ideal $T$ of $S$, $(T) = T$.
6. $[A] \Gamma [B] \subseteq [A \Gamma B]$, for all subsets $A, B$ of $S$.
7. $[A \cup B] = [A] \cup [B]$, for all subsets $A, B$ of $S$.
8. If $A$ and $B$ are two $\Gamma$-ideal of $S$, then the union $A \cup B$ is a $\Gamma$-ideal of $S$.

**Lemma 2.3.** Let $S$ be an ordered LA-Γ-semigroup and $A_i$ be a $\Gamma$-ideal of $S$ for all $i \in I$. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i$ is a $\Gamma$-ideal of $S$. 

Example 3.1. Let $A$ be a non-empty subset of an ordered LA-$\Gamma$-semigroup $S$. The intersection of all $\Gamma$-ideals of $S$ containing $A$ is the smallest $\Gamma$-ideal of $S$ generated by $A$ and is denoted by $(A)_T$.

Lemma 2.4. Let $A$ be a non-empty subset of an ordered LA-$\Gamma$-semigroup $S$ with left identity $e$. Then $(A)_T = (A \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S)$.

Proof: Straightforward.

Corollary 2.1. Let $S$ be an ordered LA-$\Gamma$-semigroup with left identity. Then $(S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S)$ is a $\Gamma$-ideal of $S$ for all $b \in S$.

3. Two-Sided Bases of Ordered LA-$\Gamma$-Semigroups. We begin this section with the definition of two-sided bases of an ordered LA-$\Gamma$-semigroup with left identity as follows.

Definition 3.1. Let $S$ be an ordered LA-$\Gamma$-semigroup with left identity. A non-empty subset $A$ of $S$ is called a two-sided base of $S$ if it satisfies the following two conditions.

1. $S = (A \cup S\Gamma a \cup a\Gamma S \cup (S\Gamma a)\Gamma S)$.
2. If $B$ is a subset of $A$ such that $S = (B \cup S\Gamma b \cup b\Gamma S \cup (S\Gamma b)\Gamma S)$, then $B = A$.

Example 3.1. Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma\}$ with multiplication defined by

\[
\begin{array}{c|ccccc}
\gamma & a & b & c & d & c \\
\hline
a & a & a & a & a & a \\
b & a & b & c & d & c \\
c & a & c & b & c & c \\
d & c & a & d & c & c \\
c & c & c & d & c & b \\
\end{array}
\]

and $\leq = \{(a, a), (b, b), (c, c), (d, d), (c, c), (a, b), (a, c), (a, d), (a, c)\}$. Then $S$ is an ordered LA-$\Gamma$-semigroup with left identity $b$. We have the two-sided bases of $S$ are $A_1 = \{b\}, A_2 = \{c\}, A_3 = \{d\}$ and $A_4 = \{e\}$. However, $A_5 = \{a\}$ is not a two-sided base.

Example 3.2. Let $S = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha\}$ with multiplication defined by

\[
\begin{array}{c|ccccc}
\alpha & a & b & c & d & c \\
\hline
a & a & a & a & a & a \\
b & a & b & b & b & b \\
c & a & b & d & c & c \\
d & c & a & c & d & c \\
c & c & c & c & d & d \\
\end{array}
\]

and $\leq = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b)\}$. Then $S$ is an ordered LA-$\Gamma$-semigroup with left identity $d$. We have the two-sided bases of $S$ are $A_1 = \{c\}, A_2 = \{d\}$ and $A_3 = \{e\}$. However, $A_4 = \{a\}$ and $A_5 = \{b\}$ are not a two-sided bases.

To characterize when a non-empty subset of ordered LA-$\Gamma$-semigroup $S$ with left identity is a two-sided base of the ordered LA-$\Gamma$-semigroup $S$ with left identity we need the quasi-ordering defined as follows.

Definition 3.2. Let $S$ be an ordered LA-$\Gamma$-semigroup. We define a quasi-ordering on $S$ for any $a, b \in S$, $a \leq_I b$ if $(a)_T \subseteq (b)_T$.

We write $a \leq_I b$ if $a \leq_I b$ but $a \neq b$, i.e., $a_T \subset b_T$.

The following example shows that the order $\leq_I$ defined above is not, in general, a partial order.
Example 3.3. From Example 3.2, we have that \((c) \subseteq (d)\) (i.e., \(c \leq_I d\)) and \((d) \subseteq (c)\) (i.e., \(d \leq_I c\)), but \(c \neq d\). Thus, \(\leq_I\) is not a partial order on \(S\).

Lemma 3.1. Let \(S\) be an ordered \(\Lambda\)-\(\Gamma\)-semigroup. For any \(a, b \in S\), if \(a \leq b\), then \(a \leq_I b\).

Proof: Let \(a, b \in S\) such that \(a \leq b\). We will show that \(a \leq_I b\), i.e., \((a) \subseteq (b)\).

Let \(x \in (a)\). Since \(x \in (a)\), \(x \in \Sigma a \cup \Sigma \Gamma a \cup \Sigma b \cup \Sigma \Gamma b\). There are four cases to consider.

Case 1: \(y = a\). Then \(x \leq a \leq b\), so \(x \leq b\) where \(x \in b\). We have that \(x \in (b)\). Thus, \(x \in (b)\).

Case 2: \(x \in (a)\). Then \(x = s_\gamma a\) for some \(s \in S\), \(\gamma \in \Gamma\). Since \(a \leq b\), then \(a \leq s_\gamma b\).

Case 3: \(x \in (a)\). Then \(x = a_\gamma s\) for some \(s \in S\), \(\gamma \in \Gamma\). Since \(a \leq b\), then \(a \leq b_\gamma\).

Case 4: \(x \in (a)\). Then \(x = a_\gamma s\) for some \(s \in S\), \(\gamma \in \Gamma\). Since \(a \leq b\), then \(a \leq b_\gamma\).

Therefore, \(a \leq_I b\). □

Lemma 3.2. Let \(A\) be a two-sided base of an ordered \(\Lambda\)-\(\Gamma\)-semigroup \(S\) with left identity and let \(a, b \in A\). If \((a) \subseteq (b)\), then \(a \leq b\).

Proof: Assume that \(a, b \in A\) such that \((a) \subseteq (b)\), and suppose that \(a \neq b\). Let \(B = A \setminus \{a\}\). Since \(a \neq b\), \(b \in B\). To show that \((a) \subseteq (b)\), we let \(x \in (a)\). Since \(a \neq b\), \(x \in (a)\).

There are two cases to consider.

Case 1: \(z \in A\). Then \(z = s_\gamma c\) for some \(c \in A\). Since \(c \neq b\), \(c \leq_I b\).

Case 2: \(z \in (a)\). Then \(z = s_\gamma c\) for some \(c \in A\). Since \(c \neq b\), \(c \leq_I b\).

Case 3: \(z \in (a)\). Then \(z = s_\gamma c\) for some \(c \in A\). Since \(c \neq b\), \(c \leq_I b\).

Case 4: \(z \in (a)\). Then \(z = s_\gamma c\) for some \(c \in A\). Since \(c \neq b\), \(c \leq_I b\).

4. Main Results. In this section, the algebraic structure of an ordered \(\Lambda\)-\(\Gamma\)-semigroup with left identity containing two-sided bases will be presented.
Theorem 4.1. A non-empty subset $A$ of an ordered LA-$\Gamma$-semigroup $S$ with left identity, is a two-sided base of $S$ if and only if $A$ satisfies the following two conditions:

(1) for any $x \in S$ there exists $a \in A$ such that $x \leq_I a$;
(2) for any $a, b \in A$, if $a \neq b$, then neither $a \leq_I b$ nor $b \leq_I a$.

Proof: Assume that $A$ is a two-sided base of $S$. Then $S = (A)_T$. Let $x \in S$. Since $x \in S = (A)_T$, we have $x \leq y$ for some $y \in A \cup (S \Gamma (S a) \Gamma S \cup (S a) \Gamma S)$. There are four cases to consider.

Case 1: $y \in A$. Since $x \leq y$, by Lemma 3.1, we have that $x \leq_I y$.

Case 2: $y \in S \Gamma A$. Then $y = s \gamma a$ for some $s \in S$, $\gamma \in \Gamma$ and $a \in A$. By $y = s \gamma a \in S \Gamma a \subseteq (a)_T$, $S T y \subseteq S T (S \Gamma a) = (S T) \Gamma (S a) = (a \Gamma S) \Gamma S = (S T) \Gamma a = S \Gamma a \subseteq (a)_T$ and $(S T y) \Gamma S \subseteq (S T (a \Gamma S)) \Gamma S = (S T) \Gamma (S T (a \Gamma S)) = (S T) \Gamma (S T) = (S T) \Gamma a = S \Gamma a \subseteq (a)_T$. Then $y \in S T y \cup y \Gamma S \cup (S T y) \Gamma S \subseteq (a)_T$, and so $(y)_T = (y \cup S T y \cup y \Gamma S \cup (S T y) \Gamma S) \subseteq (a)_T$, i.e., $y \leq_I a$. Since $x \leq y$, by Lemma 3.1, we have $x \leq_I y$. So $x \leq_I y \leq_I a$. Thus $x \leq_I a$.

Case 3: $y \in A \Gamma S$. Then $y = s \gamma a$ for some $s \in A$, $\gamma \in \Gamma$ and $a \in A$. By $y = s \gamma a \in A \Gamma S \subseteq (a)_T$, $S T y \subseteq S T (A \Gamma S) = A \Gamma (S T) = (S T) \Gamma (A \Gamma S) = (S T) \Gamma a = S \Gamma a \subseteq (a)_T$. Then $y \in S T y \cup y \Gamma S \cup (S T y) \Gamma S \subseteq (a)_T$, and so $(y)_T = (y \cup S T y \cup y \Gamma S \cup (S T y) \Gamma S) \subseteq (a)_T$, i.e., $y \leq_I a$. Since $x \leq y$, by Lemma 3.1, we have $x \leq_I y$. So $x \leq_I y \leq_I a$. Thus $x \leq_I a$.

Hence the condition (1) holds. Next, let $a, b \in A$ such that $a \neq b$. Suppose $a \leq_I b$. Set $B = A \setminus \{a\}$. Then $B \cup B \cap A \subseteq L$. Let $x \in S$. By condition (1), there exists $c \in A$ such that $x \leq_I c$, i.e., $(x)_T \subseteq (c)_T$. There are two cases to consider. If $c \neq a$, then $c \in B$. So $x \in (x)_T \subseteq (c)_T \subseteq (B)_T$. If $c = a$, then $x \leq_I a \leq_I b$ and $x \leq_I b$, i.e., $(x)_T \subseteq (b)_T$. So $x \in (x)_T \subseteq (b)_T \subseteq (B)_T$. Thus, $S \subseteq (B)_T$ and so $S = (B)_T$. This is a contradiction. Hence $a \leq_I b$ is false. The case $b \leq_I a$ proved similarly. Hence the condition (2) holds.

Conversely, assume that the conditions (1) and (2) hold. We will show that $A$ is a two-sided base of $S$. To show that $S = (A)_T$, let $x \in S$, by condition (1), there exists $a \in A$ such that $x \leq_I a$. Then $x \in (x)_T \subseteq (a)_T \subseteq (A)_T$. So $S \subseteq (A)_T$ and clearly $(A)_T \subseteq S$. Thus, $S = (A)_T$. Next, to show that $A$ is a minimal subset of $S$ with the property $S = (A)_T$, let $B \subset A$ such that $S = (B)_T$. Then there exists $a \in A$ and $a \notin B$. Since $a \in A$, $a \in S = (B)_T$. We will show that $a \notin (B)$. If $a \notin (B)$, then $y \leq_I a$ for some $y \in B$, by Lemma 3.1, $a \leq_I y$. This is a contradiction. So $a \notin (B)$. Thus, $a \in (S T B \cup S T B \cup S T B) \Gamma S$. Since $a \in (S T B \cup S T B \cup S T B) \Gamma S$, we have $a \leq a$ for some $c \in S T B \cup S T B \cup S T B \Gamma S$. There are three cases to consider.

Case 1: $c \in S T B$. Then $c = s \gamma b_1$ for some $s \in S$, $\gamma \in \Gamma$ and $b_1 \in B$. Since $s \leq a$ and $c = s \gamma b_1 \in S T b_1 \subseteq b_1 \cup S T b_1 \cup b_1 \Gamma S \cup (S T b_1) \Gamma S$, $a \leq (b_1 \cup S T b_1 \cup b_1 \Gamma S \cup (S T b_1) \Gamma S) = (b_1)_T$. It follows that $(a)_T \subseteq (b_1)_T$. Thus, $a \leq_I b_1$ where $a, b_1 \in A$. This is a contradiction.

Case 2: $c \in B T S$. Then $c = b T S$ for some $s \in S$, $\gamma \in \Gamma$ and $b_2 \in B$. Since $s \leq a$ and $c = b T S \subseteq b_2 \cup S T b_2 \cup b_2 \Gamma S \cup (S T b_2) \Gamma S$, $a \leq (b_2 \cup S T b_2 \cup b_2 \Gamma S \cup (S T b_2) \Gamma S) = (b_2)_T$. It follows that $(a)_T \subseteq (b_2)_T$. Thus, $a \leq_I b_2$ where $a, b_2 \in A$. This is a contradiction.

Case 3: $c \in (S T B) \Gamma S$. Then $c = (s \gamma b_3) \Gamma S$ for some $s_1, s_2 \in S$, $\gamma, \beta \in \Gamma$ and $b_3 \in B$. Since $s \leq a$ and $c = (s_1 \gamma b_3) \Gamma S \subseteq (s_1 \gamma b_3) \Gamma S \subseteq b_3 \cup S T b_3 \cup b_3 \Gamma S \cup (S T b_3) \Gamma S$, $a \leq (b_3 \cup
\[ \text{STb}_3 \cup b_3 \Gamma S \cup (\text{STb}_2 \Gamma S) = (b_3)_T. \] It follows that \((a)_T \subseteq (b_3)_T\). Thus, \(a \leq_I b_3\) where \(a, b_3 \in A\). This is a contradiction.

Therefore, \(A\) is a two-sided base of \(S\). The proof is completed. \(\square\)

**Theorem 4.2.** Let \(A\) be a two-sided base of an ordered \(\Lambda\Gamma\)-semigroup \(S\) with left identity, such that \((a)_T = (b)_T\), for some \(a\) in \(A\) and \(b\) in \(S\). If \(a \neq b\), then \(S\) contains at least two two-sided bases.

**Proof:** Assume that \(a \neq b\). Suppose that \(b \in A\). Since \(a \neq b\) and \(a \in (a)_T = (b)_T = (b \cup \text{STb} \cup b \Gamma S \cup (\text{STb}) \Gamma S) = (b) \cup (\text{STb} \cup b \Gamma S \cup (\text{STb}) \Gamma S), a \in (b) \) or \(a \in (\text{STb} \cup b \Gamma S \cup (\text{STb}) \Gamma S). \) If \(a \in (b)\), then \(a \leq b\), by Lemma 3.1, we have \(a \leq_I b\) where \(a, b \in A\). This is a contradiction. So \(a \in (\text{STb} \cup b \Gamma S \cup (\text{STb}) \Gamma S). \) By Lemma 3.2, \(a = b\). This is a contradiction.

Thus, \(b \in S \setminus A\). Setting \(B = (A \setminus \{a\}) \cup \{b\}\), then \(B \neq A\). We will show that \(B\) is a two-sided base of \(S\) using Theorem 4.1. First, let \(x \in S\). Since \(A\) is a two-sided base of \(S\), by Theorem 4.1(1), \(x \leq_I c\) for some \(c \in A\). If \(c \neq a\), then \(c \in B\). If \(c = a\), then \((c)_T = (a)_T\). Since \((a)_T = (b)_T\), we have \((c)_T = (b)_T\), i.e., \(c \leq_I b\). So \(x \leq_I c \leq_I b\). Thus, \(x \leq_I b\) where \(b \in B\). Next, let \(c_1, c_2 \in B\) such that \(c_1 \neq c_2\). We will show that neither \(c_1 \leq_I c_2\) nor \(c_2 \leq_I c_1\). Then there are four cases to consider.

Case 1: \(c_1 \neq b\) and \(c_2 \neq b\). Then \(c_1, c_2 \in A\). Since \(A\) is a two-sided base of \(S\), then neither \(c_1 \leq_I c_2\) nor \(c_2 \leq_I c_1\).

Case 2: \(c_1 \neq b\) and \(c_2 = b\). Then \((c_2)_T = (b)_T\). If \(c_1 \leq_I c_2\), then \((c_1)_T \subseteq (c_2)_T = (b)_T = (a)_T\). Thus, \(c_1 \leq_I a\) where \(c_1, a \in A\). This is contradiction. If \(c_2 \leq_I c_1\), then \((a)_T = (b)_T = (c_2)_T \subseteq (c_1)_T\). Thus, \(a \leq_I c_1\) where \(c_1, a \in A\). This is a contradiction.

Case 3: \(c_1 = b\) and \(c_2 \neq b\). Then \((c_1)_T = (b)_T\). If \(c_1 \leq_I c_2\), then \((a)_T = (b)_T = (c_1)_T \subseteq (c_2)_T\). Thus, \(a \leq_I c_2\) where \(c_2, a \in A\). This is contradiction. If \(c_2 \leq_I c_1\), then \((c_2)_T \subseteq (c_1)_T = (b)_T = (a)_T\). Thus, \(c_2 \leq_I a\) where \(c_2, a \in A\). This is a contradiction.

Case 4: \(c_1 = b\) and \(c_2 = b\). This is impossible.

Therefore, \(B\) is a two-sided base of \(S\). \(\square\)

The following corollary follows directly from Theorem 4.2.

**Corollary 4.1.** Let \(A\) be a two-sided base of an ordered \(\Lambda\Gamma\)-semigroup \(S\) with left identity, and let \(a \in A\). If \((x)_T = (a)_T\) for some \(x \in S\), \(x \neq a\), then \(x\) belongs to some two-sided base of \(S\), which is different from \(A\).

**Theorem 4.3.** Let \(A\) and \(B\) be two-sided bases of ordered \(\Lambda\Gamma\)-semigroup \(S\) with left identity. Then \(A\) and \(B\) have the same cardinality.

**Proof:** Let \(A\) and \(B\) be two-sided bases of \(S\). Let \(a \in A\). Since \(B\) is a two-sided base of \(S\), by Theorem 4.1(1), there exists \(b \in B\) such that \(a \leq_I b\). Similarly, since \(A\) is a two-sided base of \(S\), there exists \(a^* \in A\) such that \(b \leq_I a^*\). So \(a \leq_I b \leq_I a^*\) and \(a \leq_I a^*\). By Theorem 4.1(2), \(a = a^*\). Hence \((a)_T \neq (b)_T\). Now, define a mapping \(\varphi : A \to B; \varphi(a) = b\) for all \(a \in A\). First, to show that \(\varphi\) is well-defined, let \(a_1, a_2 \in A\) such that \(a_1 = a_2\), \(\varphi(a_1) = b_1\), and \(\varphi(a_2) = b_2\) for some \(b_1, b_2 \in B\). Then \((a_1)_T = (b_1)_T\) and \((a_2)_T = (b_2)_T\). Since \(a_1 = a_2\), \((a_1)_T = (a_2)_T\). Thus, \((a_1)_T = (a_2)_T = (b_1)_T = (b_2)_T\), so \(b_1 \leq_I b_2\) and \(b_2 \leq_I b_1\). By Theorem 4.1(2), \(b_1 = b_2\). Hence \(\varphi(a_1) = \varphi(a_2)\). Therefore, \(\varphi\) is well-defined. Next, to show that \(\varphi\) is one-to-one, let \(a_1, a_2 \in A\) such that \(\varphi(a_1) = \varphi(a_2)\). Then \(\varphi(a_1) = \varphi(a_2) = b\) for some \(b \in B\). We have \((a_1)_T = (a_2)_T = (b)_T\). Since \((a_1)_T = (a_2)_T, a_1 \leq_I a_2\) and \(a_2 \leq_I a_1\). Thus, \(a_1 = a_2\). Therefore, \(\varphi\) is one-to-one. Finally, to show that \(\varphi\) is onto, let \(b \in B\), and then there exists \(a \in A\) such that \(b \leq_I a\). Similarly, there exists \(b^* \in B\) such that \(a \leq_I b^*\). Then \(b \leq_I a \leq_I b^*, i.e., b \leq_I b^*\). By Theorem 4.1(2), \(b = b^*\). So \(b \leq_I a\) and \(a \leq_I b\), i.e., \((b)_T = (a)_T\) and \((a)_T = (b)_T\). Thus, \((a)_T = (b)_T\). Therefore, \(\varphi\) is onto. This completes the proof. \(\square\)

If a two-sided base of an ordered \(\Lambda\Gamma\)-semigroup \(S\) with left identity, is a \(\Gamma\)-ideal of \(S\), then \(S = (A \cup \text{ST}A \cup A \Gamma S \cup (\text{ST}A) \Gamma S) \subseteq (A \cup A \cup A \cup A) = (A) = A\). Hence \(S = A\). The
converse statement is obvious. Then we conclude that a two-sided base $A$ of an ordered LA-$\Gamma$-semigroup $S$ with left identity, is a $\Gamma$-ideal of $S$ if and only if $A = S$.

In Example 3.1, it is observed that not every two-sided base of an ordered LA-$\Gamma$-semigroup $S$ with left identity, is an LA-$\Gamma$-subsemigroup. The following theorem gives necessary and sufficient conditions of a two-sided base of an ordered LA-$\Gamma$-semigroup $S$ with left identity, to be an LA-$\Gamma$-subsemigroup.

**Theorem 4.4.** A two-sided base $A$ of an ordered LA-$\Gamma$-semigroup $S$ with left identity, is an LA-$\Gamma$-subsemigroup if and only if $A = \{a\}$ with $a\gamma a = a$ for all $\gamma \in \Gamma$.

**Proof:** Assume that $A$ is an LA-$\Gamma$-subsemigroup of $S$. Let $a, b \in A$ and $\gamma \in \Gamma$. Since $A$ is an LA-$\Gamma$-subsemigroup $S$, we have $a\gamma b \in A$. Set $a\gamma b = c$. Then $c = a\gamma b \in STb \subseteq (STb \cup b\Gamma S \cup STb\Gamma S)$. By Lemma 3.2, we have $c = b$. So $a\gamma b = b$. Similarly, $c = a\gamma b \in a\Gamma S \subseteq (STa \cup a\Gamma S \cup STa\Gamma S)$. By Lemma 3.2, we have $c = a$. So $a\gamma b = a$. Thus, $a = b$. Therefore, $A = \{a\}$ with $a\gamma a = a$. The converse statement is clear. □

The union of all two-sided bases of an ordered LA-$\Gamma$-semigroup $S$ with left identity is denoted by $C$.

**Theorem 4.5.** Let $S$ be an ordered LA-$\Gamma$-semigroup with left identity. Then $S \setminus C = \emptyset$ or a $\Gamma$-ideal of $S$.

**Proof:** Assume that $S \setminus C \neq \emptyset$. We will show that $S \setminus C$ is a $\Gamma$-ideal of $S$. Let $x \in S$, $\gamma \in \Gamma$ and $a \in S \setminus C$. To show that $x\gamma a \in S \setminus C$ and $a\gamma x \in S \setminus C$, suppose that $x\gamma a \not\in S \setminus C$. Then $x\gamma a \in C$. Thus, $x\gamma a \in A$ for a two-sided base $A$ of $S$. Let $x\gamma a = b$ for some $b \in A$. Since $b = x\gamma a \in STa \subseteq (a)_T$, $b \subseteq (a)_T$. It follows that $(b)_T \subseteq (a)_T$. If $(b)_T = (a)_T$, by Corollary 4.1, we have that $a \in C$. This is a contradiction. Thus, $(b)_T \subset (a)_T$, i.e., $b <_I a$. Since $A$ is a two-sided base of $S$, by Theorem 4.1(1), there exists $b_1 \in A$ such that $a < b_1$. Since $b <_I a$, $b_1 <_I b_1$, $b \leq b_1$ where $b, b_1 \in A$. This is a contradiction. Thus, $x\gamma a \in S \setminus C$. Similarly, we can show that $a\gamma x \in S \setminus C$. Next, to show that if $a_1 \in S \setminus C$ and $a_2 \in S$ such that $a_2 \leq a_1$, then $a_2 \in S \setminus C$. Suppose that $a_2 \in C$. Then $a_2 \in B$ for a two-sided base $B$ of $S$. Since $B$ is a two-sided base of $S$, by Theorem 4.1(1), there exists $a_3 \in B$ such that $a_1 \leq a_3$. Since $a_2 \leq a_1$, by Lemma 3.1, $a_2 \leq a_1$. We have that $a_2 < a_3$ where $a_2, a_3 \in B$. This is a contradiction. Thus, $a_2 \not\in C$, i.e., $a_2 \in S \setminus C$. Therefore, $S \setminus C$ is a $\Gamma$-ideal of $S$.

Let $M^*$ be a proper $\Gamma$-ideal of an ordered LA-$\Gamma$-semigroup $S$ with left identity, containing every proper $\Gamma$-ideal of $S$.

**Theorem 4.6.** Let $S$ be an ordered LA-$\Gamma$-semigroup with left identity, and $\emptyset \neq C \subseteq S$. Then $S \setminus C = M^*$ if and only if every two-sided base of $S$ is one-element base.

**Proof:** Assume that $S \setminus C = M^*$. Then $S \setminus C$ is a maximal proper $\Gamma$-ideal of $S$. We will show that for every $a \in C$, $C \subseteq (a)_T$. Let $a \in C$. Suppose $C \not\subseteq (a)_T$. Since $C \not\subseteq (a)_T$ and $\emptyset \neq C \subseteq S$, $(a)_T$ is a proper $\Gamma$-ideal of $S$. Thus, $a \in (a)_T \subseteq M^* = S \setminus C$, and so $a \not\in C$, i.e., $a \not\in C$. This is a contradiction. Hence $C \subseteq (a)_T$ for every $a \in C$. We will show that for every $a \in C$, $S \setminus C \subseteq (a)_T$. Suppose that $S \setminus C \not\subseteq (a)_T$ for some $a^* \in C$. Then $(a^*)_T \not= S$, and so $(a^*)_T$ is a proper $\Gamma$-ideal of $S$. Thus, $a^* \in (a^*)_T \subseteq M^* = S \setminus C$, and so $a^* \in S \setminus C$, i.e., $a^* \not\in C$. This is a contradiction. Hence $S \setminus C \subseteq (a)_T$ for every $a \in C$. Since $S \setminus C \subseteq (a)_T$ and $C \subseteq (a)_T$ for every $a \in C$, we have $S = (S \setminus C) \cup C \subseteq (a)_T \subseteq S$. So $S = (a)_T$ for every $a \in C$. Thus, $(a)$ is a two-sided base of $S$. Next, let $A$ be a two-sided base of $S$. We will show that $a = b$ for every $a, b \in A$. Suppose that there exists $a, b \in A$ such that $a \neq b$. Since $A$ is a two-sided base of $S$, $a \in A \subseteq C$ and $a \in C$. So $S = (a)_T$. Since $a \neq b$ and $b \in S = (a \cup STa \cup a\Gamma S \cup (STa)\Gamma S) = (a) \cup (STa \cup a\Gamma S \cup (STa)\Gamma S)$, $b \in (a)$ or $b \in (STa \cup a\Gamma S \cup (STa)\Gamma S)$. If $b \in (a)$, then $b \leq a$ by Lemma 3.1, $b \leq a$. This is a contradiction. So $b \in (STa \cup a\Gamma S \cup (STa)\Gamma S)$. By Lemma 3.2, $a = b$. This is a contradiction. Therefore, every two-sided base of $S$ is one-element base.
Theorem 4.5.\  S/C \subseteq C \subseteq S. By Theorem 4.6, we obtain Theorem 4.8.\  \Gamma\text{-}\text{ideal of } A \text{ can study other results in this algebraic structures. Moreover, we may use the essential } \Gamma\text{-semigroup with left identity is the maximal proper } \Gamma\text{-ideal. In the future work, we in Theorem 4.8 that the complement of union of all two-sided base of an ordered LA-}

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Conversely, assume that every two-sided base of S is a one-element base. Then S = (a)\_T for every a \in C. To show that S\backslash C = M^*\, since \varnothing \neq C \subset S, \varnothing \neq S\backslash C \subset S. By Theorem 4.5, S/C is a proper \Gamma\text{-}\text{ideal of } S. Next, let M be a proper \Gamma\text{-}\text{ideal of } S such that S\backslash C \subset M \subset S. Since S\backslash C \subset M, there exists x \in M such that x \notin S\backslash C, i.e., x \in C. We have x \in M \cap C. So M \cap C \neq \varnothing. Let b \in M \cap C. Since b \in M, STb \subseteq S\Gamma M \subseteq M, &bF \subseteq MF \subseteq M and (STb)F \subseteq (S\Gamma M)F \subseteq MF \subseteq M, b \in STb \cup bF \subseteq (STb)F \subseteq M. We have (b)\_T = (b \cup STb \cup bF \cup (STb)F) \subseteq \{M\} = M. Since b \in C, by assumption, we have (b)\_T = S. So S = (b)\_T \subseteq M \subset S. Thus, M = S. This is a contradiction. Hence S\backslash C is a maximal proper \Gamma\text{-}\text{ideal of } S. Finally, let B be a \Gamma\text{-}\text{ideal of } S such that B \varsubsetneq S\backslash C. Since B \varsubsetneq S\backslash C, there exists x \in B such that x \notin S\backslash C, i.e., x \in C. So B \cap C \neq \varnothing. Let c \in B \cap C. Since c \in B, STc \subseteq S\Gamma B \subseteq B, c\Gamma S \subseteq B\Gamma S \subseteq B and (STc)\Gamma S \subseteq (STB)\Gamma S \subseteq B\Gamma S \subseteq B, c \subseteq STc \cup c\Gamma S \cup (STc)\Gamma S \subseteq B. We have (c)\_T = (c \cup STc \cup c\Gamma S \cup (STc)\Gamma S) \subseteq \{B\} = B. Since c \in C, S = (c)\_T \subseteq B \subseteq S. Thus, S = B. Therefore, S\backslash C = M^*.

Theorem 4.7. Let S be an ordered LA-\Gamma\text{-}\text{semigroup with left identity. If } e \text{ is a left identity of } S, then } \{e\} \text{ is a two-sided base of } S.

Proof: Assume that e is a left identity of S. Let A = \{e\}. We will show that A is a two-sided base of S. To show that S = (A)\_T, since e is a left identity of S, by Lemma 2.1, we have S = e\Gamma S = S\Gamma e. Since S = S\Gamma e, we have (S\Gamma e)\Gamma S = (S\Gamma e)\Gamma (S\Gamma e) = (S\Gamma S)\Gamma (e\Gamma e) = S\Gamma e. So e \cup STe \subseteq e\Gamma S \cup (STe)\Gamma = S. Thus, (A)\_T = (e \cup STe \subseteq e\Gamma S \cup (STe)\Gamma) \subseteq S. Hence (A)\_T = S. Clearly, A is a minimal subset of S with the property S = (A)\_T. Therefore, A is a two-sided base of S.

In Examples 3.1 and 3.2, it is observed that every two-sided base of an ordered LA-\Gamma\text{-}\text{semigroup with left identity is one-element base. This leads to proving the following corollary. From Theorem 4.3 and Theorem 4.7, we can easily obtain the following result.

Corollary 4.2. Let S be an ordered LA-\Gamma\text{-}\text{semigroup with left identity. Then every two-sided base of } S \text{ is one-element base.}

In Example 3.2, we have the all two-sided bases of S are A_1 = \{c\}, A_2 = \{d\} and A_3 = \{e\}. Then S\backslash C = \{a, b\} is a maximal proper \Gamma\text{-}\text{ideal of } S containing every proper \Gamma\text{-}\text{ideal of } S. We have the following result is combining Theorem 4.6 and Corollary 4.2.

Theorem 4.8. Let S be an ordered LA-\Gamma\text{-}\text{semigroup with left identity. Then } S\backslash C \text{ is a maximal proper } \Gamma\text{-}\text{ideal of } S containing all proper } \Gamma\text{-}\text{ideals of } S.

Proof: Let S be an ordered LA-\Gamma\text{-}\text{semigroup with left identity. By Corollary 4.2, we have every two-sided base of } S \text{ is one-element base. Since every two-sided base of } S \text{ is one element base, by Theorem 4.6, we obtain } S\backslash C = M^*. Therefore, S\backslash C \text{ is a maximal proper } \Gamma\text{-}\text{ideal of } S \text{ containing all proper } \Gamma\text{-}\text{ideals of } S.

5. Conclusion. In this paper, we focus on the results for two-sided bases of ordered LA-\Gamma\text{-}\text{semigroups with left identity. We show in Corollary 4.2 that every two-sided base of an ordered LA-\Gamma\text{-}\text{semigroup with left identity is one-element base. Finally, we prove in Theorem 4.8 that the complement of union of all two-sided base of an ordered LA-\Gamma\text{-}\text{semigroup with left identity is the maximal proper } \Gamma\text{-}\text{ideal. In the future work, we can study other results in this algebraic structures. Moreover, we may use the essential } (m, n)\text{-}\text{ideal of semigroups defined in [10] to define essential } (m, n)\text{-}\text{bases of semigroups and study their properties.}

REFERENCES


