ICIC Express Letters
Volume **16**, Number **7**, July **2022**

ICIC International ©2022 ISSN 1881-803X
pp. 713–721

ON LEFT AND RIGHT BASES OF LA-Γ-SEMIHYPERGROUPS WITH PURE LEFT IDENTITY

Wichayaporn Jantanan[1], Ronnason Chinram[2] and Saranya Hangsawat[3,*]

[1]Department of Mathematics
Faculty of Science
Buriram Rajabhat University
Nai Muang, Muang, Buriram 31000, Thailand
wichayaporn.jan@bru.ac.th

[2]Division of Computational Science
Faculty of Science
Prince of Songkla University
Hat Yai, Songkhla 90110, Thailand
ronnason.c@psu.ac.th

[3]Mathematics Program
Faculty of Science and Technology
Songkhla Rajabhat University
Muang, Songkhla 90000, Thailand
*Corresponding author: saranya.nu@skru.ac.th

Received August 2021; accepted October 2021

Abstract. *In this paper, we introduce the concepts of left and right bases of LA-*Γ*-semihypergroups with pure left identity and study the structure of LA-*Γ*-semihypergroups with pure left identity containing left and right bases. We focus only on the results for right base of an LA-*Γ*-semihypergroup with pure left identity. For left base, we can show dually. We also give the necessary and sufficient condition for element in an LA-*Γ*-semihypergroup with pure left identity, to be a right base. Moreover, we show that all right bases of an LA-*Γ*-semihypergroup with pure left identity have the same cardinality. Finally, we show that the compliment of the union of all right bases of an LA-*Γ*-semihypergroup with pure left identity is maximal proper left* Γ*-hyperideal.*
Keywords: LA-Γ-semihypergroup, Left Γ-hyperideal, Right base, Quasi-order, Maximal proper left Γ-hyperideal

1. **Introduction.** The algebraic hyperstructure notion was introduced in 1934 by Marty [1]. The attraction of hyperstructure is its special property that the image of each pair of a cross product of two sets is led to a set where in classical structures it is an element again, as follows.

Let S be a non-empty set and $P^*(S) = P(S) \setminus \{\varnothing\}$ denotes the set of all non-empty subsets of S. The map $\circ$: $S \times S \to P^*(S)$ is called a hyperoperation or join operation on the set S. A couple $(S, \circ)$ is called a hypergroupoid. Let A and B be two non-empty subsets of S, and then we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \quad a \circ A = \{a\} \circ A \text{ and } a \circ B = \{a\} \circ B.$$

In [2], Sen introduced the notion of a Γ-semigroup as a generalization of semigroups and ternary semigroups, as follows.

DOI: 10.24507/icicel.16.07.713

Let S and Γ be any two non-empty sets. Then S is called a *Γ-semigroup* if there is a mapping from $S \times \Gamma \times S$ into S, written as $(a, \alpha, b) \mapsto a\alpha b$, such that $(a\gamma b)\beta c = a\gamma(b\beta c)$ for all $a, b, c \in S$ and all $\gamma, \beta \in \Gamma$.

In 1955, the notion of a right (left) base of a semigroups was first introduced by Tamura [3]. Later, Fabrici [4] studied the structure of semigroups containing the right bases by using Tamura's results. Recently, the notions of left and right bases of Γ-semigroups were introduced by Changphas and Kummoon [5]. In this paper, we introduce the concepts of left and right bases of LA-Γ-semihypergroups with pure left identity. In particular, we study the structure of LA-Γ-semihypergroups with pure left identity containing the right bases and extend the results in Γ-semigroups to LA-Γ-semihypergroups. This structure was defined by Yaqoob and Aslam [8] which is a generalization of many algebraic structures, for example, commutative Γ-semigroups, LA-semigroups, comutative semihypergroups and LA-semihypergroups. They received some nice results in LA-Γ-semihypergroups.

2. **Preliminaries.** In this section, we provide definitions and results that are used throughout this paper. Those can be found in [6, 7, 8, 9].

Definition 2.1. *Let S and Γ be any two non-empty sets. Then S is called a* left almost Γ-semihypergroup *(LA-Γ-semihypergroup) if every $\gamma \in \Gamma$ is a hyperoperation on S, i.e., $x\gamma y \subseteq S$, for every $x, y \in S$. And for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have*

$$(x\alpha y)\beta z = (z\alpha y)\beta x.$$

The law $(x\alpha y)\beta z = (z\alpha y)\beta x$ is called left invertive law. For A and B be two non-empty subsets of an LA-Γ-semihypergroup S, we define

$$A\gamma B = \bigcup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$$

also

$$A\Gamma B = \bigcup_{\gamma\in\Gamma} A\gamma B = \bigcup\{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

Throughout the paper, S stands for an LA-Γ-semihypergroup unless otherwise specified.

Suggest that the notion of LA-Γ-semihypergroups is a generalization of commutative semigroups, commutative semihypergroups and of commutative Γ-semigroups.

Example 2.1. [8] *Let $S = \{1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$ be the sets of binary hyperoperations defined below:*

α	1	2	3
1	$\{1,3\}$	$\{1,2\}$	$\{1,3\}$
2	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$
3	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

β	1	2	3
1	$\{1,3\}$	$\{1,2,3\}$	$\{1,3\}$
2	$\{1,2,3\}$	$\{1,2\}$	$\{2,3\}$
3	$\{1,2,3\}$	$\{2,3\}$	$\{1,2\}$

Clearly S is not a Γ-semihypergroup because $\{1, 2, 3\} = (1\alpha 1)\beta 3 \neq 1\alpha(1\beta 3) = \{1, 3\}$. Thus, S is an LA-Γ-semihypergroup because it satisfies the left invertive law.

Every LA-Γ-semihypergroup satisfies the law $(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$ for all $a, b, c, d \in S$ and $\alpha, \beta, \gamma \in \Gamma$. This law is known as Γ-hypermedial law [8].

Definition 2.2. *Let S be an LA-Γ-semihypergroup. An element $e \in S$ is called a* left identity (resp. pure left identity) *if $a \in e\gamma a$ (resp. $a = e\gamma a$) for all $a \in S$ and $\gamma \in \Gamma$.*

By Example 2.1, elements 1 and 2 in S are left identities of S but not pure left identity of S.

Example 2.2. *Let $S = \{x_1, x_2, x_3, x_4\}$ and $\Gamma = \{\beta\}$ be the sets of binary hyperoperations defined below:*

β	x_1	x_2	x_3	x_4
x_1	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$
x_2	$\{x_3\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	$\{x_4\}$
x_3	$\{x_2\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	$\{x_4\}$
x_4	$\{x_4\}$	$\{x_4\}$	$\{x_4\}$	$\{x_4\}$

Clearly S is not a Γ-semihypergroup because $\{x_2\} = (x_2\beta x_1)\beta x_1 \neq x_2\beta(x_1\beta x_1) = \{x_3\}$. Thus, S is an LA-Γ-semihypergroup because it satisfies the left invertive law. Here x_1 is a left identity of S; moreover x_1 is a pure left identity of S.

Lemma 2.1. *Let S be an LA-Γ-semihypergroup with pure left identity e, then $(a\alpha b)\beta(c\gamma d) = (d\alpha c)\beta(b\gamma a)$ holds for all $a, b, c, d \in S$ and $\alpha, \beta, \gamma \in \Gamma$.*

Proof: Let S be an LA-Γ-semihypergroup with pure left identity e. Then for all $a, b, c, d \in S$ and $\alpha, \beta, \gamma \in \Gamma$, we have

$$\begin{aligned}(a\alpha b)\beta(c\gamma d) &= ((e\gamma a)\alpha b)\beta((e\alpha c)\gamma d)\\ &= ((b\gamma a)\alpha e)\beta((d\alpha c)\gamma e) \quad \text{(by left invertive law)}\\ &= ((b\gamma a)\alpha(d\alpha c))\beta(e\gamma e) \quad \text{(by } \Gamma\text{-hypermedial law)}\\ &= ((e\gamma e)\alpha(d\alpha c))\beta(b\gamma a) \quad \text{(by left invertive law)}\\ &= (e\alpha(d\alpha c))\beta(b\gamma a)\\ &= (d\alpha c)\beta(b\gamma a)\end{aligned}$$

This completes the proof. □

The law $(a\alpha b)\beta(c\gamma d) = (d\alpha c)\beta(b\gamma a)$ is called a Γ-hyperparamedial law.

Lemma 2.2. *If S is an LA-Γ-semihypergroup with pure left identity e, then $S\Gamma S = S$.*

Proof: Clearly, $S\Gamma S \subseteq S$. Next, to show that $S \subseteq S\Gamma S$. Let $x \in S$, and then for any $\gamma \in \Gamma$, we have $x = e\gamma x \subseteq S\Gamma S$. Thus, $S \subseteq S\Gamma S$. Hence, $S\Gamma S = S$. □

Definition 2.3. *Let S be an LA-Γ-semihypergroup.*

(1) *A non-empty subset A of S is called a* sub LA-Γ-semihypergroup *of S if $x\gamma y \subseteq A$ for all $x, y \in A$ and $\gamma \in \Gamma$.*

(2) *A non-empty subset A of S is called a* left (resp. right) Γ-hyperideal *of S if $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$).*

(3) *A left Γ-hyperideal A of S is called* proper *if $A \neq S$.*

(4) *A proper left Γ-hyperideal A of S is called* maximal *if for any left Γ-hyperideal B of S such that $A \subseteq B$ implies $A = B$ or $B = S$.*

Lemma 2.3. *Let S be an LA-Γ-semihypergroup and A_i be a left Γ-hyperideal of S for each $i \in I$, and then the following statements hold.*

(1) *If $\bigcap_{i\in I} A_i \neq \varnothing$, then $\bigcap_{i\in I} A_i$ is a left Γ-hyperideal of S.*

(2) *$\bigcup_{i\in I} A_i$ is a left Γ-hyperideal of S.*

Proof: (1) Assume that $\bigcap_{i\in I} A_i \neq \varnothing$. To show that $S\Gamma\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} A_i$, let $x \in S\Gamma\left(\bigcap_{i\in I} A_i\right)$. Then $x \in s\gamma a_1$ for some $s \in S$, $\gamma \in \Gamma$ and $a_1 \in \bigcap_{i\in I} A_i$. Since $a_1 \in \bigcap_{i\in I} A_i$, we obtain $a_1 \in A_i$ for all $i \in I$. Since A_i is a left Γ-hyperideal of S for all $i \in I$, we have $x \in s\gamma a_1 \subseteq S\Gamma A_i \subseteq A_i$, for all $i \in I$. So $x \in \bigcap_{i\in I} A_i$. Hence, $S\Gamma\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} A_i$. Therefore, $\bigcap_{i\in I} A_i$ is a left Γ-hyperideal of S.

(2) To show that $\bigcup_{i\in I} A_i$ is a left Γ-hyperideal of S, let $x \in S\Gamma\left(\bigcup_{i\in I} A_i\right)$. Then $x \in s\gamma a_1$ for some $s \in S$, $\gamma \in \Gamma$ and $a_1 \in \bigcup_{i\in I} A_i$. Since $a_1 \in \bigcup_{i\in I} A_i$, we obtain $a_1 \in A_i$ for some $i \in I$. Since A_i is a left Γ-hyperideal of S for all $i \in I$, $x \in s\gamma a_1 \subseteq S\Gamma A_i \subseteq A_i \subseteq \bigcup_{i\in I} A_i$. Thus, $x \in \bigcup_{i\in I} A_i$. Hence, $S\Gamma\left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I} A_i$. Therefore, $\bigcup_{i\in I} A_i$ is a left Γ-hyperideal of S. □

Definition 2.4. *Let A be a non-empty subset of an LA-Γ-semihypergroup S. The intersection of all left Γ-hyperideals of S containing A, is the* smallest left Γ-hyperideal *of S generated by A and is denoted by $(A)_L$.*

Lemma 2.4. *Let A be a non-empty subset of LA-Γ-semihypergroup S with pure left identity e. Then*

$$(A)_L = A \cup S\Gamma A.$$

Proof: Let $B = A \cup S\Gamma A$. First, consider

$$\begin{aligned} S\Gamma B &= S\Gamma(A \cup S\Gamma A) \\ &= S\Gamma A \cup S\Gamma(S\Gamma A) \\ &= S\Gamma A \cup (S\Gamma S)\Gamma(S\Gamma A) && \text{(by Lemma 2.2)} \\ &= S\Gamma A \cup (A\Gamma S)\Gamma(S\Gamma S) && \text{(by } \Gamma\text{-hyperparamedial law)} \\ &= S\Gamma A \cup (A\Gamma S)\Gamma S \\ &= S\Gamma A \cup (S\Gamma S)\Gamma A && \text{(by left invertive law)} \\ &= S\Gamma A \cup S\Gamma A = S\Gamma A \subseteq B. \end{aligned}$$

Thus, B is a left Γ-hyperideal of S containing A. Next, let C be a left Γ-hyperideal of S containing A. We obtain $A \subseteq C$, and so $S\Gamma A \subseteq S\Gamma C \subseteq C$. Thus, $B = A \cup S\Gamma A \subseteq C$. Hence, B is the smallest left Γ-hyperideal of S containing A. Therefore, $(A)_L = A \cup S\Gamma A$. □

3. **Main Results.** We begin this section with the definition of a right base of an LA-Γ-semihypergroup with pure left identity as follows.

Definition 3.1. *Let S be an LA-Γ-semihypergroup with pure left identity. A non-empty subset A of S is called a* right base *of S if it satisfies the following two conditions.*
(1) *$S = A \cup S\Gamma A$, i.e., $S = (A)_L$.*
(2) *If B is a subset of A such that $S = (B)_L$, then $B = A$.*
For a left base *of S it is defined dually.*

By Example 2.2, S is an LA-Γ-semihypergroup with pure left identity. Then we have $\{x_1\}$ as the only one right base of S.

Example 3.1. *Let $S = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ be the sets of binary hyperoperations defined below:*

γ	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{d\}$
c	$\{b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	S

Clearly S is not a Γ-semihypergroup because $\{b\} = (b\gamma a)\gamma a \neq b\gamma(a\gamma a) = \{c\}$. Thus, S is an LA-Γ-semihypergroup because it satisfies the left invertive law and S is an LA-Γ-semihypergroup with pure left identity. Then, the right bases of S are $A = \{a\}$, $B = \{b\}$, $C = \{c\}$ and $D = \{d\}$. And the left bases of S are the same as the right bases of S.

Lemma 3.1. *Let A be a right base of an LA-Γ-semihypergroup S with pure left identity and $a, b \in A$. If $a \in S\Gamma b$, then $a = b$.*

Proof: Assume that $a \in S\Gamma b$ and suppose that $a \neq b$. Let $B = A \setminus \{a\}$, then $B \subset A$. Since $a \neq b$, $b \in B$. To show that $(A)_L \subseteq (B)_L$, let $x \in (A)_L = A \cup S\Gamma A$. Then $x \in A$ or $x \in S\Gamma A$. Let $x \in A$. If $x \neq a$, then $x \in B \subseteq B \cup S\Gamma B$. So $x \in (B)_L$. If $x = a$ by assumption we have $x = a \in S\Gamma b \subseteq S\Gamma B \subseteq B \cup S\Gamma B$. So $x \in (B)_L$. Hence, $A \subseteq (B)_L$. Let $x \in S\Gamma A$. Then $x \in s\gamma c$ for some $s \in S$, $\gamma \in \Gamma$ and $c \in A$. If $c \neq a$, then $c \in B$. So

$x \in s\gamma c \subseteq S\Gamma B \subseteq B \cup S\Gamma B = (B)_L$. Thus, $x \in (B)_L$. If $c = a$, then $c = a \in S\Gamma b \subseteq S\Gamma B$ So $x \in s\gamma c \subseteq S\Gamma(S\Gamma B)$

$$\begin{aligned} &= (S\Gamma S)\Gamma(S\Gamma B) && \text{(by Lemma 2.2)} \\ &= (B\Gamma S)\Gamma(S\Gamma S) && \text{(by } \Gamma\text{-hyperparamedial law)} \\ &= (B\Gamma S)\Gamma S \\ &= (S\Gamma S)\Gamma B && \text{(by left invertive law)} \\ &= S\Gamma B \subseteq (B)_L. \end{aligned}$$

Thus, $S\Gamma A \subseteq (B)_L$. Since $A \subseteq (B)_L$ and $S\Gamma A \subseteq (B)_L$, $(A)_L = A \cup S\Gamma A \subseteq (B)_L$. By $S = (A)_L \subseteq (B)_L \subseteq S$, so we obtain $(B)_L = S$. This contradicts to condition (2) of Definition 2.1. Therefore, $a = b$. □

Let S be an LA-Γ-semihypergroup with pure left identity. Define a quasi-order on S by, for any $a, b \in S$,

$$a \leq_L b \Leftrightarrow (a)_L \subseteq (b)_L.$$

We write $a <_L b$ if $a \leq_L b$ but $a \neq b$, i.e., $(a)_L \subset (b)_L$.

In general, $\leq_L$ is not a partial order. By Example 3.1, we have $(a)_L \subseteq (b)_L$, i.e., $a \leq_L b$ and $(b)_L \subseteq (a)_L$, i.e., $b \leq_L a$ but $a \neq b$. This shows that $\leq_L$ is not a partial order.

The following theorem characterizes when a non-empty subset of an LA-Γ-semihypergroup with pure left identity, is a right base of an LA-Γ-semihypergroup with pure left identity.

Theorem 3.1. *A non-empty subset A of an LA-Γ-semihypergroup S with pure left identity, is a right base if and only if A satisfies the following two conditions:*

(1) *for any $x \in S$, there exists $a \in A$ such that $x \leq_L a$;*

(2) *for any $a, b \in A$, if $a \neq b$, then neither $a \leq_L b$ nor $b \leq_L a$.*

Proof: Assume that A is a right base of S. Then $S = (A)_L$. First, let $x \in S$, and then $x \in S = (A)_L = A \cup S\Gamma A$. We have $x \in A$ or $x \in S\Gamma A$. If $x \in A$, then $x \leq_L x$. If $x \in S\Gamma A$, then $x \in s\gamma a$ for some $s \in S$, $\gamma \in \Gamma$ and $a \in A$. Since $x \in s\gamma a \subseteq S\Gamma a \subseteq (a)_L$, $x \in (a)_L$. Since $x \in S\Gamma a$, $S\Gamma x \subseteq S\Gamma(S\Gamma a)$

$$\begin{aligned} &= (S\Gamma S)\Gamma(S\Gamma a) && \text{(by Lemma 2.2)} \\ &= (a\Gamma S)\Gamma(S\Gamma S) && \text{(by } \Gamma\text{-hyperparamedial law)} \\ &= (a\Gamma S)\Gamma S \\ &= (S\Gamma S)\Gamma a && \text{(by left invertive law)} \\ &= S\Gamma a \subseteq (a)_L. \end{aligned}$$

We obtain $S\Gamma x \subseteq (a)_L$. Since $x \subseteq (a)_L$ and $S\Gamma x \subseteq (a)_L$, $(x)_L = x \cup S\Gamma x \subseteq (a)_L$. So $x \leq_L a$. Hence, the condition (1) holds. Next, let $a, b \in A$ be such that $a \neq b$. Suppose that $a \leq_L b$, and then $(a)_L \subseteq (b)_L$. Since $a \in (a)_L \subseteq (b)_L$ and $a \neq b$, $a \in S\Gamma b$, by Lemma 3.1, $a = b$. This is a contradiction. The case $b \leq_L a$ can be proved similarly. Thus, $a \leq_L b$ and $b \leq_L a$ are false. Hence, the condition (2) holds.

Conversely, assume that (1) and (2) hold. We will show that A is a right base of S. First, to show that $S = (A)_L$, let $x \in S$, by (1) there exists $a \in A$ such that $(x)_L \subseteq (a)_L$, then $x \in (x)_L \subseteq (a)_L \subseteq (A)_L$. So $S \subseteq (A)_L$, and $S = (A)_L$. Next, to show that A is a minimal subset of S with the property $S = (A)_L$. Let $B \subset A$ such that $S = (B)_L$. Since $B \subset A$, there exists $a \in A$ and $a \notin B$. Since $a \in A \subseteq S = (B)_L$ and $a \notin B$, we obtain $a \in S\Gamma B$. Then $a \in s\gamma b$ for some $s \in S$, $\gamma \in \Gamma$ and $b \in B$. By $a \in s\gamma b \subseteq S\Gamma b \subseteq (b)_L$, so $a \in (b)_L$. Since $a \in S\Gamma b$, $S\Gamma a \subseteq S\Gamma(S\Gamma b)$

$$\begin{aligned} &= (S\Gamma S)\Gamma(S\Gamma b) && \text{(by Lemma 2.2)} \\ &= (S\Gamma b)\Gamma(S\Gamma S) && \text{(by } \Gamma\text{-hyperparamedial law)} \\ &= (S\Gamma b)\Gamma S \end{aligned}$$

$$= (S\Gamma S)\Gamma b \qquad \text{(by left invertive law)}$$
$$= S\Gamma b \subseteq (b)_L.$$

By $a \subseteq (b)_L$ and $S\Gamma a \subseteq (b)_L$, so $(a)_L = a \cup S\Gamma a \subseteq (b)_L$. Thus, $a \leq_L b$ where $a, b \in A$. This contradicts to the condition (2). Therefore, A is a right base of S. □

If a right base A of an LA-Γ-semihypergroup S with pure left identity, is a left Γ-hyperideal of S, then

$$S = A \cup S\Gamma A \subseteq A \cup A = A.$$

Hence, $S = A$. The converse statement is obvious. Then we conclude as the following.

Theorem 3.2. *A right base A of an LA-Γ-semihypergroup S with pure left identity, is a left Γ-hyperideal of S if and only if $A = S$.*

Definition 3.2. *An LA-Γ-semihypergroup S is said to be a right singular if $y \in x\gamma y$ for all $x, y \in S$ and $\gamma \in \Gamma$.*

Theorem 3.3. *Let A be a right base of an LA-Γ-semihypergroup S with pure left identity. If A is a sub LA-Γ-semihypergroup of S, then A is right singular.*

Proof: Assume that A is a sub LA-Γ-semihypergroup of S. Let $a, b \in A$ and let $\gamma \in \Gamma$. By assumption, $a\gamma b \subseteq A$. Set $c \in a\gamma b$ for some $c \in A$. Since $c \in a\gamma b \subseteq S\Gamma b$, by Lemma 3.1, we have $c = b$. Thus, $b \in a\gamma b$. Therefore, A is right singular. □

The converse statement is not valid in general. By Example 3.1, we have $B = \{b\}$ as a right base of S such that B is right singular. Then B is not sub LA-Γ-semihypergroup of S because $B\Gamma B = \{a, b, c\} \nsubseteq B$.

Let S be an LA-Γ-semihypergroup, and let $\alpha \in \Gamma$. An element e of S is called an α-idempotent of S if $e \in e\alpha e$. Let $E_\alpha(S)$ denote the set of all α-idempotent of S, and let $E(S) = \bigcup_{\alpha\in\Gamma} E_\alpha(S)$.

By Theorem 3.3, we obtain the following corollary.

Corollary 3.1. *Let A be a right base of an LA-Γ-semihypergroup S with pure left identity. If A is a sub LA-Γ-semihypergroup of S, then $E(S) \neq \varnothing$.*

Proof: Assume that A is a sub LA-Γ-semihypergroup of S. Let $e \in A$ and let $\alpha \in \Gamma$. Then $e\alpha e \subseteq A$. By Theorem 3.3, we obtain $e \in e\alpha e$. Thus, e is an α-idempotent of S. Therefore, $E(S) \neq \varnothing$. □

Theorem 3.4. *The right bases of an LA-Γ-semihypergroup S with pure left identity have the same cardinality.*

Proof: Let A and B be right bases of S. Let $a \in A$. Since B is a right base of S, by Theorem 3.1(1), there exists $b \in B$ such that $a \leq_L b$. Similarly, since A is a right base of S, there exists $a' \in A$ such that $b \leq_L a'$. Then $a \leq_L b \leq_L a'$, and $a \leq_L a'$. By Theorem 3.1(2), $a = a'$. Hence, $(a)_L = (b)_L$. Now, define a mapping

$$\varphi : A \to B; \quad \varphi(a) = b$$

for all $a \in A$. First, to show that φ is well-defined, let $a_1, a_2 \in A$ be such that $a_1 = a_2$, $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$ for some $b_1, b_2 \in B$. Then $(a_1)_L = (b_1)_L$ and $(a_2)_L = (b_2)_L$. Since $a_1 = a_2$, $(a_1)_L = (a_2)_L$, so $(a_1)_L = (a_2)_L = (b_1)_L = (b_2)_L$. Thus, $b_1 \leq_L b_2$ and $b_2 \leq_L b_1$. By Theorem 3.1(2), $b_1 = b_2$. Hence, $\varphi(a_1) = \varphi(a_2)$. Therefore, φ is well-defined.

Next, to show that φ is one-to-one, let $a_1, a_2 \in A$ be such that $\varphi(a_1) = \varphi(a_2)$. Then $\varphi(a_1) = \varphi(a_2) = b$ for some $b \in B$. So, we obtain $(a_1)_L = (a_2)_L = (b)_L$. Since $(a_1)_L = (a_2)_L$, $a_1 \leq_L a_2$ and $a_2 \leq_L a_1$. By Theorem 3.1(2), $a_1 = a_2$. Therefore, φ is well-defined.

Finally, to show that φ is onto, let $b \in B$. Since A is a right base of S, by Theorem 3.1(1), there exists $a \in A$ such that $b \leq_L a$. Since B is a right base of S, by Theorem 3.1(1) there exists $b' \in B$ such that $a \leq_L b'$. So, we obtain $b \leq_L a \leq_L b'$ and $b \leq_L b'$. By

Theorem 3.1(2), $b = b'$. Thus, $(a)_L = (b)_L$. Hence, $\varphi(a) = b$. Therefore, φ is onto. This completes the proof. □

Theorem 3.5. *Let A be a right base of an LA-Γ-semihypergroup S with pure left identity and let $a \in A$. If $(a)_L = (b)_L$ for some $b \in S$ such that $a \neq b$, then b is an element of a right base of S which is different from A.*

Proof: Assume that $(a)_L = (b)_L$ for some $b \in S$ such that $a \neq b$. Let $B = (A\backslash\{a\})\cup\{b\}$, and then $B \neq A$. We will show that B is a right base of S. To show that B satisfies (1) in Theorem 3.1, let $x \in S$. Since A is a right base of S, by Theorem 3.1(1) there exists $c \in A$ such that $x \leq_L c$. If $c \neq a$, then $c \in B$. Thus, $x \leq_L c$ where $c \in B$. If $c = a$, then $(c)_L = (a)_L$. Since $(a)_L = (b)_L$, $(c)_L = (b)_L$. Thus, $(x)_L \subseteq (c)_L = (b)_L$. Hence, $x \leq_L b$ where $b \in B$. Next, to show that B satisfies (2) in Theorem 3.1, let $b_1, b_2 \in B$ be such that $b_1 \neq b_2$. Then there are four cases to consider.

Case 1: $b_1 \neq b$ and $b_2 \neq b$. Then $b_1, b_2 \in A$. Since A is a right base of S, neither $b_1 \leq_L b_2$ nor $b_2 \leq_L b_1$.

Case 2: $b_1 \neq b$ and $b_2 = b$. Then $b_1 \in A\backslash\{a\}$ and $(b_2)_L = (b)_L$. If $b_1 \leq_L b_2$, then $(b_1)_L \subseteq (b_2)_L = (b)_L = (a)_L$. So $b_1 \leq_L a$ where $b_1, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $(a)_L = (b)_L = (b_2)_L \subseteq (b_1)_L$. So $a \leq_L b_1$ where $b_1, a \in A$. This is a contradiction.

Case 3: $b_1 = b$ and $b_2 \neq b$. Then $(b_1)_L = (b)_L$ and $b_2 \in A\backslash\{a\}$. If $b_1 \leq_L b_2$, then $(a)_L = (b)_L = (b_1)_L \subseteq (b_2)_L$. So $a \leq_L b_2$ where $b_2, a \in A$. This is a contradiction. If $b_2 \leq_L b_1$, then $(b_2)_L \subseteq (b_1)_L = (b)_L = (a)_L$. So $b_2 \leq_L a$ where $b_2, a \in A$. This is a contradiction.

Case 4: $b_1 = b$ and $b_2 = b$. This is impossible.

Therefore, B is a right base of S which $B \neq A$. □

Theorem 3.6. *Let A^* be the union of all right bases of an LA-Γ-semihypergroup S with pure left identity. If $S\backslash A^* \neq \varnothing$, then $S\backslash A^*$ is a left Γ-hyperideal of S.*

Proof: Assume that $S\backslash A^* \neq \varnothing$. We will show that $S\backslash A^*$ is a left Γ-hyperideal of S. Let $x \in S$, $\gamma \in \Gamma$ and $a \in S\backslash A^*$. To show that $x\gamma a \subseteq S\backslash A^*$, suppose that $x\gamma a \nsubseteq S\backslash A^*$. Then there exist $b \in x\gamma a$ and $b \notin S\backslash A^*$, i.e., $b \in A^*$. So $b \in A$ for some a right base A of S. Since $b \in x\gamma a \subseteq S\Gamma a \subseteq (a)_L$, $S\Gamma b \subseteq S\Gamma(S\Gamma a)$

$$
\begin{aligned}
&= (S\Gamma S)\Gamma(S\Gamma a) && \text{(by Lemma 2.2)}\\
&= (a\Gamma S)\Gamma(S\Gamma S) && \text{(by } \Gamma\text{-hyperparamedial law)}\\
&= (a\Gamma S)\Gamma S\\
&= (S\Gamma S)\Gamma a && \text{(by left invertive law)}\\
&= S\Gamma a \subseteq (a)_L.
\end{aligned}
$$

So $(b)_L = b \cup S\Gamma b \subseteq (a)_L$. Thus, $(b)_L \subseteq (a)_L$. If $(b)_L = (a)_L$, by Theorem 3.5, we obtain $a \in A^*$. This is a contradiction. Hence, $(b)_L \subset (a)_L$, i.e., $b <_L a$. Since A is a right base of S by Theorem 3.1(1), there exists $b_1 \in A$ such that $a \leq_L b_1$. Then $b <_L a \leq_L b_1$ and so $b \leq_L b_1$ where $b, b_1 \in A$. This contradicts to the condition (2) of Theorem 3.1. Hence, $x\gamma a \subseteq S\backslash A^*$. Therefore, $S\backslash A^*$ is a left Γ-hyperideal of S. □

In Example 2.2, we have the union of all right base of S as $A^* = \{x_1\}$. Then $S \setminus A^* = \{x_2, x_3, x_4\}$ is a maximal proper left Γ-hyperideal of S. However, it turns out that this is true in general, when $A^* \neq S$ and $A^* \subseteq (a)_L$ for all $a \in A^*$. Then we will prove in Theorem 3.7.

Theorem 3.7. *Let S be an LA-Γ-semihypergroup with pure left identity and let A^* be the union of all right bases of S such that $A^* \neq \varnothing$. Then $S\backslash A^*$ is a maximal proper left Γ-hyperideal of S if and only if $A^* \neq S$ and $A^* \subseteq (a)_L$ for all $a \in A^*$.*

Proof: Let $S\backslash A^*$ be a maximal proper left Γ-hyperideal of S. Then $A^* \neq S$. Let $a \in A^*$. Suppose that $A^* \nsubseteq (a)_L$. Then there exist $x \notin (a)_L$ and $x \in A^*$, i.e., $x \notin S\backslash A^*$. We have $(S\backslash A^*) \cup (a)_L \subset S$. So $(S\backslash A^*) \cup (a)_L$ is a proper left Γ-hyperideal of S. This contradicts to the maximality of $S\backslash A^*$. Therefore, $A^* \subseteq (a)_L$.

Conversely, let $A^* \neq S$ and $A^* \subseteq (a)_L$ for all $a \in A^*$. Then, we obtain $\varnothing \neq A^* \subset S$, $\varnothing \neq S\backslash A^* \subset S$. By Theorem 3.6, $S\backslash A^*$ is a proper left Γ-hyperideal of S. Let L be a left Γ-hyperideal of S such that $S\backslash A^* \subseteq L \subseteq S$. Suppose that $S\backslash A^* \neq L$, and so $S\backslash A^* \subset L$. Then there exist $x \in L$ and $x \notin S\backslash A^*$, i.e., $x \in A^*$. So $L \cap A^* \neq \varnothing$. Let $a \in L \cap A^*$. Then $a \in L$ and $S\Gamma a \subseteq S\Gamma L \subseteq L$. So $(a)_L = a \cup S\Gamma a \subseteq L$. Since $(a)_L \subseteq L$, $A^* \subseteq (a)_L$ and $S\backslash A^* \subset L$. We have $S = (S\backslash A^*) \cup A^* \subseteq L \cup (a)_L \subseteq L \subseteq S$. Thus, $S = L$. Therefore, $S\backslash A^*$ is a maximal proper left Γ-hyperideal of S. □

Theorem 3.8. *Let S be an LA-Γ-semihypergroup with pure left identity and let A^* be the union of all right bases of S such that $\varnothing \neq A^* \subset S$. If S contains a maximal left Γ-hyperideal of S containing every proper left Γ-hyperideal of S, denoted by L^*, then $S\backslash A^* = L^*$ if and only if $|A| = 1$ for every right base A of S.*

Proof: Assume that $S\backslash A^* = L^*$. Then $S\backslash A^*$ is a maximal proper left Γ-hyperideal of S. By Theorem 3.7, $A^* \subseteq (a)_L$ for all $a \in A^*$. We will show that $S\backslash A^* \subseteq (a)_L$ for all $a \in A^*$. Suppose that $S\backslash A^* \nsubseteq (a')_L$ for some $a' \in A^*$. Then $(a')_L \subset S$, and $(a')_L$ is a proper left Γ-hyperideal of S. So $a' \in (a')_L \subseteq L^* = S\backslash A^*$ and we obtain $a' \in S\backslash A^*$, i.e., $a' \notin A^*$. This is a contradiction. Hence, $S\backslash A^* \subseteq (a)_L$ for all $a \in A^*$. By $S = (S\backslash A^*) \cup A^* \subseteq (a)_L \subseteq S$ for all $a \in A^*$. So $S = (a)_L$ for all $a \in A^*$. Therefore, $\{a\}$ is a right base of S for all $a \in A^*$. Let A be a right base of S and let $a, b \in A$. Suppose that $a \neq b$. Since $A \subseteq A^*$, $a \in A^*$ and so $S = (a)_L$. Since $a \neq b$ and $b \in S = a \cup S\Gamma a$, we obtain $b \in S\Gamma a$. By Lemma 3.1, $b = a$. This is a contradiction. Thus, $a = b$ and $|A| = 1$.

Conversely, assume that every right base of S has only one element. Then $S = (a)_L$ for all $a \in A^*$. We will show that $S\backslash A^* = L^*$. Since $\varnothing \neq A^* \subset S$, $\varnothing \neq S\backslash A^* \subset S$. By Theorem 3.6, $S\backslash A^*$ is a proper left Γ-hyperideal of S. Let L be a left Γ-hyperideal of S such that $S\backslash A^* \subseteq L \subseteq S$. Suppose that $S\backslash A^* \neq L$, so $S\backslash A^* \subset L$. Then there exist $x \in L$ and $x \notin S\backslash A^*$, i.e., $x \in A^*$. So $L \cap A^* \neq \varnothing$. Let $a \in L \cap A^*$. Since $a \in L$, we have $S\Gamma a \subseteq S\Gamma L \subseteq L$. So $S = a \cup S\Gamma a \subseteq L \subseteq S$. Thus, $L = S$. Hence, $S\backslash A^*$ is a maximal proper left Γ-hyperideal of S. Next, let B be a proper left Γ-hyperideal of S. If $B \nsubseteq S\backslash A^*$, then there exist $a \in B$ and $a \notin S\backslash A^*$, i.e., $a \in A^*$. Since $a \in B$, we have $S\Gamma a \subseteq S\Gamma B \subseteq B$. So $S = a \cup S\Gamma a \subseteq B \subset S$. Thus, $S = B$. This is a contradiction. Hence, $B \subseteq S\backslash A^*$. Therefore, $S\backslash A^* = L^*$ and the proof is completed. □

We end this paper with an example by illustrating the results of Theorem 3.8.

By Example 2.2, we have the union of all right base of S as $A^* = \{x_1\}$. Then $S\backslash A^* = \{x_2, x_3, x_4\}$ is a maximal left Γ-hyperideal of S containing every proper left Γ-hyperideal of S. So, we obtain $S\backslash A^* = L^*$ and $|\{x_1\}| = 1$.

4. **Conclusion.** In this paper, we focus only on the results for right base of an LA-Γ-semihypergroup with pure left identity. For left base, we can show dually. In Theorem 3.1, we give the necessary and sufficient condition for element in an LA-Γ-semihypergroup with pure left identity, to be a right base. In Theorem 3.4, we show that all right bases of an LA-Γ-semihypergroup with pure left identity have the same cardinality. Moreover, we show the remarkable results of an LA-Γ-semihypergroup with pure left identity in Theorems 3.2, 3.3, 3.5, 3.6, 3.7 and 3.8. In the future work, we can study other results in this algebraic hyperstructures. For example in [10], the authors studied the fuzzy almost interior ideals in semigroups, and we can extend this result to the fuzzy almost interior hyperideals in LA-Γ-semihypergroups.

REFERENCES

[1] F. Marty, On a generalization of the notion of groups, *Proc. of the 8th Congress of Scandinavian Mathematicians*, Stockholm, Sweden, pp.45-49, 1934.

[2] M. K. Sen, On Γ-semigroups, *Proc. of the International Conference on Algebra and Its Application*, Decker Publication, New York, pp.301-308, 1981.

[3] T. Tamura, One sided-bases and translation of a semigroup, *Math. Japan.*, vol.3, pp.137-141, 1955.

[4] I. Fabrici, One-sided bases of semigroups, *Matematicky Casopis*, vol.22, pp.286-290, 1972.

[5] T. Changphas and P. Kummoon, On left and right bases of a Γ-semigroup, *International Journal of Pure and Applied Mathematicians*, vol.118, pp.125-135, 2018.

[6] M. Aslam, T. Aroob and N. Yaqoob, On cubic Γ-hyperideals in left almost Γ-semihypergroups, *Annals of Fuzzy Mathematics and Informatics*, vol.5, no.1, pp.169-182, 2013.

[7] N. Yaqoob, Applications of rough sets to Γ-hyperideals in left almost Γ-semihypergroups, *Neural Computing and Applications*, vol.21, no.1, pp.267-273, 2012.

[8] N. Yaqoob and M. Aslam, On bi-Γ-hyperideals in left almost Γ-semihypergroups, *Journal of Advanced Research in Pure Mathematics*, vol.4, no.1, pp.130-143, 2012.

[9] N. Yaqoob, M. Aslam and Faisal, On soft Γ-hyperideals over left almost Γ-semihypergroups, *Journal of Advanced Research in Dynamical and Control Systems*, vol.4, no.1, pp.1-12, 2012.

[10] W. Krailoet, A. Simuen, R. Chinram and P. Petchkaew, A note on fuzzy almost interior ideals in semigroups, *International Journal of Mathematics and Computer Science*, vol.16, no.2, pp.803-808, 2021.