ICIC Express Letters
Volume **16**, Number **8**, August **2022**

ICIC International ©2022 ISSN 1881-803X
pp. 835–840

BRANCH-AND-PRICE FOR SPLIT DELIVERY VEHICLE ROUTING PROBLEM

Kosuke Tezuka[1], Yanli Huo[2], Chunhui Xu[3] and Takayuki Shiina[1,*]

[1]Department of Industrial and Management Systems Engineering
Waseda University
3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
tz-ann0225.ntbkr@suou.waseda.jp; *Corresponding author: tshiina@waseda.jp

[2]College of Economics and Management
China Jiliang University
No. 258, Xueyuan Street, Xiasha Higher Education Zone, Hangzhou 310018, P. R. China
hyl091029@yahoo.co.jp

[3]Faculty of Social Systems Science
Chiba Institute of Technology
2-17-1, Tsudanuma, Narashino-Shi, Chiba 275-0016, Japan
joh.haruki@gmail.com

Received December 2021; accepted February 2022

Abstract. *In the basic vehicle routing problem (VRP), a customer's demand should be satisfied by only one vehicle. On the other hand, split delivery vehicle routing problem (SDVRP) allows two or more vehicles to serve each customer. SDVRP has more feasible solutions than basic VRP, making it more difficult to enumerate feasible solutions. Column generation is a well-known method for VRP, and we can also efficiently solve SDVRP using Dantzig-Wolf decomposition. For integer programming, branch-and-price, which is a combination of column generation and branch-and-bound, is effective. In this research, we adopt branch-and-price for the solution method, and discuss the model and the solution process as an exact solution method for SDVRP. In addition, we compare the results of experiments for two solution methods and evaluate their relative performance.*
Keywords: Integer programming, Vehicle routing problem, Column generation, Split delivery, Labeling algorithm

1. **Introduction.** Many organizations, especially transportation companies, must move their products or services to designated places to meet customer demand. Since transportation costs are often large, it is important to create an efficient operations schedule. However, the schedule must satisfy certain constraints, such as the condition that delivery vehicles have limited loading capacity. These constraints make it difficult to determine an optimal plan. This problem is known as the vehicle routing problem (VRP). While basic VRP assumes that the demand of a certain customer should be satisfied by only one vehicle, split delivery vehicle routing problem (SDVRP) allows a customer demand to be served by two or more vehicles.

VRP is often solved by the cutting plane method, or branch-and-cut, as is also SD-VRP [1, 2]. These researchers focus on how cuts can be added effectively, but it is very difficult to define the cut addition scheme, for reasons mainly related to subtour elimination constraints. The purpose of this research is to develop a labeling algorithm. Column generation is also a well-known method for VRP, because it simplifies the model. Dantzig-Wolfe decomposition reduces VRP or SDVRP to a simplex master problem, and a subproblem which can be considered as the shortest path problem with resource constraints. We can obtain the solution of a subproblem without subtour by applying dynamic programming.

DOI: 10.24507/icicel.16.08.835

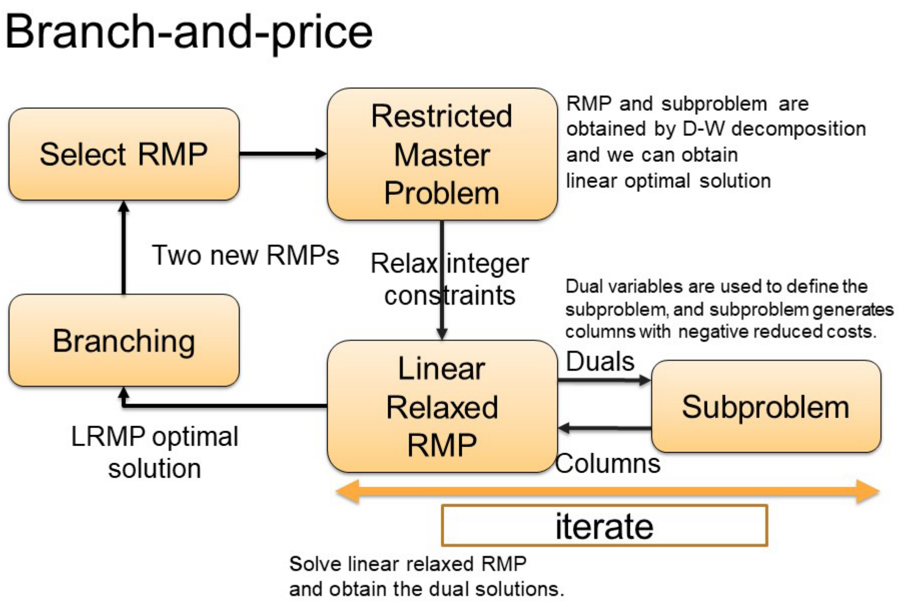

FIGURE 1. Framework of branch-and-price

Actually, many researchers using column generation have been successful, for example [3]. Branch-and-price, the combination of column generation and branch-and-bound, is effective for integer programming. The framework of branch-and-price is shown in Figure 1. In the present research, we have used the formulation and the branch-and-price algorithm for SDVRP, as shown in [4, 5]. Desaulniers et al. [6] provided a survey overview in integer programming column generation and its many applications.

The organization of this paper is as follows. Section 1 introduces SDVRP, and Section 2 shows the formulation and solution algorithm we develop. Section 3 shows the results of the numerical experiments, and the final section gives the conclusions.

2. Formulation of SDVRP.

2.1. **Master problem.** All vehicles leave from the start depot, visit a set of customers and satisfy each customer's demand. Finally, the vehicles arrive at the end depot. In this process, split deliveries are allowed. "Route" represents the path traversed from the start depot to the end depot. A sufficient number of vehicles is available, having the same limited capacity. We assume an arc cost which equals the distance and satisfies the triangle inequality. "Delivery pattern" distinguishes patterns on the same route by the amount of product that the vehicles deliver to each customer.

Sets

N	Set of customers, $\{1, \ldots, n\}$
P	Set of points, including N, the start and the end depot $\{0, n+1\}$
A	Set of arcs
R	Set of routes, combinations of arcs from the start depot to the end depot
W_r	Set of delivery patterns on a route r

Parameters

c_r	Cost of route r
Q	Capacity of vehicle
d_i	Demand of customer i
δ_{irw}	Delivery quantity for customer i by delivery pattern w on route r
b_{ijr}	If route r contains arc (i,j) then 1, otherwise 0

Decision Variables

- θ_{rw} The number of vehicles executing delivery pattern w of route r
- θ_r The number of vehicles on route r
- y_{ij} The number of vehicles passing on arc (i, j)
- l_i The number of vehicles leaving customer i
- H The number of vehicles used in this model

In addition, some variables have upper and lower bounds. Symbols $\underline{H}$, $\overline{H}$, $\underline{l}_i$, $\bar{l}_i$, $\underline{y}_{ij}$, $\overline{y}_{ij}$ define the bounds of respective variables. If the costs of arcs satisfy triangle inequality, Dror et al. [2] proved that $\overline{y}_{ij} \leq 1$, $\forall i, j \in N$ is a valid inequality. Also, $\underline{H} = \left\lfloor \sum_{i \in N} d_i/Q \right\rfloor$ is a valid equality, while $\overline{H} = 2\left\lfloor \sum_{i \in N} d_i/Q \right\rfloor$ and $\underline{l}_i \geq 1$, $\forall i \in N$ are valid inequalities. The following is the formulation to SDVRP by column expression. The restricted master problem (RMP) is formulated as follows.

$$\min \sum_{r \in R} \sum_{w \in W_r} c_r \theta_{rw} \tag{1}$$

$$\sum_{r \in R} \sum_{w \in W_r} \delta_{irw} \theta_{rw} \geq d_i, \quad \forall i \in N \tag{2}$$

$$\sum_{r \in R} \sum_{w \in W_r} (b_{ijr} + b_{jir}) \theta_{rw} \leq 1, \quad \forall i, j \in N\colon i < j \tag{3}$$

$$\sum_{r \in R \setminus \{0\}} \sum_{w \in W_r} \theta_{rw} = H \tag{4}$$

$$\sum_{j \in P} \sum_{r \in R} \sum_{w \in W_r} b_{ijr} \theta_{rw} = l_i, \quad \forall i \in N \tag{5}$$

$$\sum_{r \in R} \sum_{w \in W_r} b_{ijr} \theta_{rw} = y_{ij}, \quad \forall (i, j) \in A \tag{6}$$

$$\underline{H} \leq H \leq \overline{H} \tag{7}$$

$$\underline{l}_i \leq l_i \leq \bar{l}_i, \quad \forall i \in N \tag{8}$$

$$\underline{y}_{ij} \leq y_{ij} \leq \overline{y}_{ij}, \quad \forall (i, j) \in A \tag{9}$$

$$\sum_{w \in W_r} \theta_{rw} = \theta_r, \quad \forall r \in R \tag{10}$$

$$\theta_{rw} \geq 0, \quad \forall r \in R, \forall w \in W_r \tag{11}$$

$$\theta_r, \text{integer}, \quad \forall r \in R \tag{12}$$

$$H, \text{integer} \tag{13}$$

$$l_i, \text{integer}, \quad \forall i \in N \tag{14}$$

$$y_{ij}, \text{integer}, \quad \forall (i, j) \in A \tag{15}$$

Objective function (1) minimizes total cost. Constraint (2) represents demand satisfaction. Inequality (3) is cutting plane, as shown in [2]. Equalities (4)-(6) define variables H, l_i, y_{ij} by θ_{rw}, and constraints (7)-(9) are the bounds. Constraint (10) indicates the link between θ_{rw} and θ_r.

2.2. **Subproblem.** Dual variables π_i, β_{ij}, η, γ_i, α_{ij} correspond to constraints (2)-(6). By solving the linear relaxation of RMP, we can obtain dual variables which define the subproblem (SP). SP minimizes the objective function that represents reduced cost, and the constraints that must be satisfied for each vehicle. SP can be solved as the shortest path problem of the graph as shown in Figure 2. SP is defined to find a new column corresponding to the new schedule.

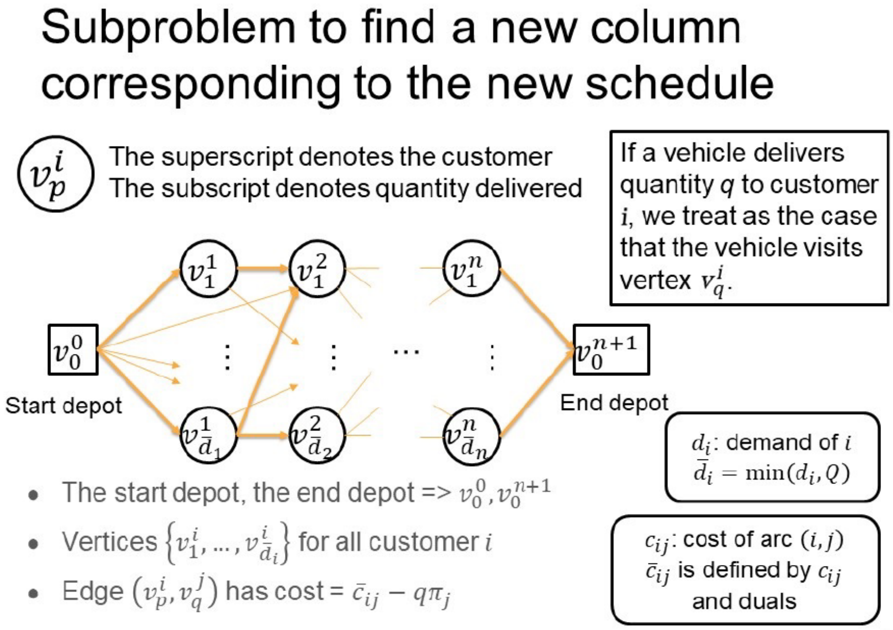

FIGURE 2. Definition of the shortest path problem

Parameters

c_{ij} Cost of arc (i,j)

$\bar{c}_{ij}$ $= c_{ij} - \beta_{ij} - \gamma_i - \alpha_{ij}$

$\bar{d}_i$ $= \min(d_i, Q)$, possible quantity of delivery for customer i

Decision Variables

x_{ij} If the vehicle passes on arc (i,j) then 1, otherwise 0

δ_i Delivery quantity for customer i by the vehicle

$$c^* = \min \sum_{(i,j)\in A} \bar{c}_{ij} x_{ij} - \sum_{i\in N} \delta_i \pi_i - \eta \tag{16}$$

$$\sum_{j\in P\backslash\{0\}} x_{0j} = 1 \tag{17}$$

$$\sum_{j\in P} x_{ij} - \sum_{j\in P} x_{ji} = 0, \quad \forall i \in N \tag{18}$$

$$\sum_{j\in P\backslash\{n+1\}} x_{j,n+1} = 1 \tag{19}$$

$$\sum_{i\in S} \sum_{j\in S\backslash\{i\}} x_{ij} \le |S| - 1, \quad S \subseteq N,\ |S| \ge 2 \tag{20}$$

$$\sum_{i\in N} \delta_i \le Q \tag{21}$$

$$\sum_{j\in P} x_{ij} \le \delta_i \le \bar{d}_i \sum_{j\in P} x_{ij}, \quad \forall i \in N \tag{22}$$

$$\delta_i, \text{integer}, \quad \forall i \in N \tag{23}$$

$$x_{ij} \in \{0,1\}, \quad \forall (i,j) \in A \tag{24}$$

In SP, we minimize the reduced cost for individual vehicles. Constraints (17)-(19) represent flow constraints. Inequalitiy (20) is called subtour elimination constraint. No vehicle is able to load product exceeding vehicle capacity (21). The left hand inequality of (22) eliminates the patterns that include a customer with zero delivery. The right hand inequality states that the delivery to each customer exceeds neither customer demand nor vehicle capacity. In addition to these constraints, each vehicle should deliver either 1 or $\bar{d}_i$ quantity, with at most one exception in which a customer is served quantity q, $(1 < q < \bar{d}_i)$. Archetti et al. [4] defined this as extreme delivery. Extreme deliveries represent extreme points of the problem. [5] showed that any feasible solution of SDVRP is given as a combination of extreme deliveries.

If the reduced cost is non-negative, the relaxed RMP is optimal. This provides a lower bound of the problem solution. The optimal linear relaxed RMP value z^*_{RLMP} and the lower bound of linear relaxed problem $\underline{z}^*_{RMP}$ is related with minimum reduced cost c^* as

$$\underline{z}^*_{RMP} = z^*_{RLMP} + \overline{H}c^* \tag{25}$$

We can regard SP as a shortest path problem with resource constraints. Therefore, we can solve SP by dynamic programming. Archetti et al. [4] solved SP in which a route that has cycles is also allowed. However, the optimal solution does not have routes with cycles, so generating such a route in SP cannot improve the lower bound efficiently. In this paper, we do not allow routes to include cycles, and we present a new method to generate a route.

We define a graph $G = (V, E)$. The set of vertices V includes the start depot vertex v_0^0, the end depot vertex v_0^{n+1} and vertices $\left\{v_1^i, v_2^i, \ldots, v_{\bar{d}_i}^i\right\}$ for each customer. If a vehicle delivers q quantity to customer i, we regard that as the case that the vehicle visits vertex v_q^i. Each edge of G has the weights based on (16). We set labels on the vertices. A label is expressed as $(\nu, \lambda, \sigma, C)$. The first element ν denotes the number of customers the vehicle has visited. The second element λ represents the quantity that the vehicle has delivered. The third element σ indicates whether the vehicle has already delivered $2 \le q \le \bar{d}_i - 1$ to customer i. The last element C is the current path cost corresponding to (16). In addition to these, a label has the list of customers already visited in the path. The label of v_0^0 is initialized as $(0, 0, 0, 0)$. At the step of extending a label of v_p^i to v_q^j, the status is updated as follows:

$$\nu := \nu + 1,\ \lambda := \lambda + q,\ \sigma := \text{if } (2 \le q \le \bar{d}_i - 1) \text{ then } \sigma + 1,\ C := C + \bar{c}_{ij} - q\pi_i$$

We can calculate the reduced cost rc from a partial path cost C of a label on the end depot by $c^* = C - \eta$. If a label $(\nu, \lambda, \sigma, C)$ does not satisfy the constraints $\nu \le n$, $\lambda \le Q$, $\sigma \le 1$, the label should be deleted.

We have developed a new rule of dominating a label. We solve SP at first, by adopting a simplified dominating rule. Among vertices of the same customer, the label $(\nu, \lambda, \sigma, C)$ dominates another label $(\nu', \lambda', \sigma', C')$ if $\nu \le \nu'$, $\lambda \le \lambda'$, $\sigma \le \sigma'$, $C \le C'$. This simplified rule provides a heuristic solution. If no routes that have negative reduced cost are generated in this rule, the strict rule is applied. In the strict rule, we consider about a label that has the set of visited customers S and another label that has the set S'. Only if these sets satisfy the constraints $S' \subseteq S$, the former label can dominate the latter label.

2.3. **Branching.** If RMP solution violates integer constraints, branching is conducted. Since there are integer constraints on H, l_i, y_{ij}, we branch these variables. First, if $\tilde{H}$ is fractional, two subproblems are generated, of which one is updated as $\overline{H} = \left\lfloor \tilde{H} \right\rfloor$, and the other as $\underline{H} = \left\lceil \tilde{H} \right\rceil$. Second, l_i is considered. In the same way as above, a subproblem has the constraint $\bar{l}_i = \left\lfloor \tilde{l}_i \right\rfloor$ or $\underline{l}_i = \left\lceil \tilde{l}_i \right\rceil$. At last, we branch by y_{ij} to $\overline{y}_{ij} = \lfloor \tilde{y}_{ij} \rfloor$ or $\underline{y}_{ij} = \lceil \tilde{y}_{ij} \rceil$. When all integer constraints are satisfied, a feasible solution is obtained. If this feasible objective function value is less than the current optimal value, it defines the lower bound of the problem solution. In this research, we selected the branching node by best-first searching. At each iteration of column generation on a tree node, the lower bound is provided by (25) or the optimal solution of linear relaxed RMP.

3. **Numerical Experiments.** We ran the algorithm on HP Z840 Workstation, Windows 10 Pro (64bit), CPU: Intel Xeon E5-2620 v3 2.4GHz, RAM: 16.0GB, using AMPL and CPLEX 12.7.1.0.

The coordinates of n customers and a depot are generated randomly on a two dimensional map. The demands follow uniform distribution, in which the minimum is 10 and the maximum is 50. We assumed the capacity $Q = 40$. Table 1 shows the results. The calculation time represents the sum of AMPL system time and the executing time of CPLEX.

Table 1. Results of the experiments

	Column generation heuristics					Branch-and-price				
n	obj.	gap	time	column	node	obj.	gap	time	column	node
5	425	0.0	0.125	13	0	425	0.0	0.125	13	1
10	1082	15.5	0.360	25	6	937	0.0	37.215	139	371
15	1078	4.6	2.771	56	13	1031	0.0	29.781	111	155
20	1728	83.9	7.145	76	103	940	0.0	219.707	293	917
25	2596	26.9	5.633	69	378	2045	0.0	621.282	501	1555
30	2247	7.4	23.829	129	519	2092	0.0	873.049	792	3349

Column generation heuristics continue to generate columns, and RMP is solved when new columns cannot be generated. Only an approximate solution is obtained by column generation heuristics; on the other hand, the exact solution can be detected by branch-and-price. Branch-and-price takes longer calculation time. In some cases, the solution of column generation heuristics can be equal to the exact solution. However, branch-and-price may be better because it is guaranteed to get exact integer solution.

4. **Concluding Remarks.** An exact solution algorithm to solve SDVRP has been developed. During test runs, it was found that branch-and-price and the labeling algorithm required long calculation time. Therefore, we need to improve the SP algorithm so as to reduce its calculation time, such as by using search heuristics. We also plan to extend the model to a time-windows model and to a multiple start or end depots model.

REFERENCES

[1] C. Archetti, N. Bianchessi and M. G. Speranza, Branch-and-cut algorithms for the split delivery vehicle routing problem, *European Journal of Operational Research*, vol.238, pp.685-698, 2014.

[2] M. Dror, G. Laporte and P. Trudeau, Split delivery routing, *Naval Research Logistics*, vol.37, pp.383-402, 1990.

[3] J. Desrosiers and M. E. Lubbecke A primer in column generation, *Column Generation*, pp.1-32, 2005.

[4] C. Archetti, N. Bianchessi and M. G. Speranza, A column generation approach for the split delivery vehicle routing problems, *Networks*, vol.58, 2011.

[5] G. Desaulniers, Branch-and-price-and-cut for the split-delivery vehicle routing problem with time windows, *Operations Research*, vol.58, pp.179-192, 2010.

[6] G. Desaulniers, J. Desrosiers and M. M. Solomon, *Column Generation*, Springer, 2005.