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TRIANGULAR SHAPED TYPE-2 FUZZY NUMBER AND UESU PRODUCT

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Abstract. *In 1975, Zadeh introduced the concept of type-2 fuzzy set to model imprecision and uncertainty. type-2 fuzzy set is a fuzzy set whose membership grade is type-1 fuzzy set. The extension of this concept to real numbers is the type-2 fuzzy number, and many researchers have devoted their efforts to this research. The authors defined a new type-2 fuzzy number and a method for disjoint t-norm operations. In this paper, we propose a method of performing disjoint t-norm operations on type-2 fuzzy numbers using interval operations.*

Keywords: Type-2 fuzzy number, T-norm, Triangular shaped type-2 fuzzy number, Uesu product

1. **Introduction.** Type-2 fuzzy set is a fuzzy set whose membership grade is type-1 fuzzy set. Type-2 fuzzy set on real numbers is type-2 fuzzy number, and many researchers have devoted their efforts to this research. The authors have conducted various studies on the arithmetic of type-2 fuzzy numbers.

As for type-2 fuzzy sets, Liu proposed an extension of type-2 fuzzy logic to α-cuts of type-2 fuzzy sets using the notion of α-planes [5]. Furthermore, Mendel and Liu defined type-2 fuzzy logic as a restricted special case of type-2 fuzzy logic represented by its α-plane [7]. Hamrawi and Coupland also defined quasi type-2 fuzzy numbers and derived their arithmetic operations [8].

We defined a new type-2 fuzzy number "Triangular Shaped Type-2 Fuzzy Numbers", and then we defined the t-norm operation on this number. In this paper, we propose a method of performing disjoint t-norm operations on type-2 fuzzy numbers using interval operations.

2. **Fuzzy Numbers.** In this paper, we denote the set of all real numbers by $\mathbb{R}$, the set of all type-1 fuzzy numbers on $\mathbb{R}$ by E^1, and the set of all type-2 fuzzy numbers on $\mathbb{R}$ by E^2. Before describing the definition of fuzzy numbers, we review some definitions as a preparation [3].

Definition 2.1. *Type-1 fuzzy set* [3].
Type-1 fuzzy set A on universal set X is

$$\mu_A : X \to [0, 1].$$

Type-1 fuzzy set A on universal set X is characterized by the membership function $\mu_A(x)$. For simplicity, we denote $\mu_A(x)$ by $A(x)$.

A whole type-1 fuzzy set on X is denoted by $\mathcal{F}^1(X)$. In particular, type-1 fuzzy set whose membership function $A(x)$ takes only 0 or 1 is a set in the usual sense, but it is called a crisp set to distinguish it from a fuzzy set.

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Definition 2.2. *α-level set* [3] *(α-cut).*

Let $A \in \mathcal{F}^1(X)$, α-level set (α-cut) of A is defined as follows:

$$[A]_\alpha = \{x \in X | A(x) \geq \alpha\}, \quad 0 \leq \alpha \leq 1.$$

Theorem 2.1. *Decomposition theorem* [3].

Let $A \in \mathcal{F}^1(X)$, the following equality holds

$$A = \bigcup_{0 \leq \alpha \leq 1} \alpha[A]_\alpha.$$

Here, αS (S is a crisp set) is type-1 fuzzy set with the following membership function:

$$\alpha S(x) = \begin{cases} \alpha, & x \in S \\ 0, & otherwise \end{cases}.$$

Definition 2.3. *Extension principle.*

For $f \colon X \to Y$, $A \in \mathcal{F}^1(X)$, $f(A) \in \mathcal{F}^1(Y)$ is defined by

$$\mu_{f(A)}(y) = \sup_{y=f(x)} A(x), \quad y \in Y.$$

In the same way, for $f \colon X_1 \times \cdots \times X_n \to Y$, $A_1 \in \mathcal{F}^1(X_1), \ldots, A_n \in \mathcal{F}^1(X_n)$, $f(A_1, \ldots, A_n) \in \mathcal{F}^1(Y)$ is defined by

$$\mu_{f(A_1,\ldots,A_n)}(y) = \sup_{y=f(x_1,\ldots,x_n)} \{A_1(x_1) \wedge \cdots \wedge A_n(x_n)\}.$$

Definition 2.4. *Type-1 fuzzy numbers.*

A fuzzy set u on $\mathbb{R}$ is called a type-1 fuzzy number if it satisfies the following conditions.

1) u is normal

$$\exists x_0 \in \mathbb{R}, \, u(x_0) = 1.$$

2) u is fuzzy convex

$$u(tx + (1-t)y) \geq u(x) \wedge u(y), \quad x, y \in \mathbb{R}, \, 0 \leq t \leq 1.$$

3) u is upper semicontinuous.

4) $\overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is bounded.

Here, the triangular type-1 fuzzy numbers $\langle\langle a, b, c\rangle\rangle$ are defined as follows.

Definition 2.5. *Triangular type-1 fuzzy numbers.*

$$\langle\langle a, b, c\rangle\rangle(x) = \left\{\frac{x-a}{b-a} \wedge \frac{x-c}{b-c}\right\} \vee 0$$

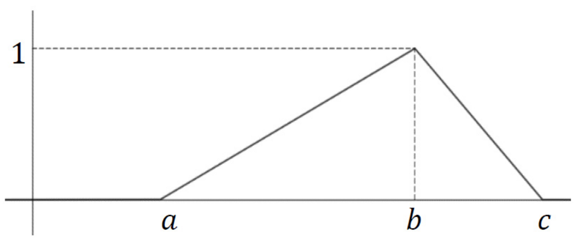

FIGURE 1. Triangular type-1 fuzzy number

Definition 2.6. *Type-2 fuzzy set* [6].

The type-2 fuzzy set $\tilde{A}$ on universal set X is

$$\mu_{\tilde{A}} : X \times J_X \to [0, 1].$$

The type-2 fuzzy set $\tilde{A}$ on universal set X is characterized by the type-2 membership function $\mu_{\tilde{A}}(x, u)$, where x and u are the primary values of $\tilde{A}$, respectively. Here, x and

u are called the primary and secondary variables of $\tilde{A}$, respectively, and J_X denotes the domain of definition of u (domain of the primary membership grade).

$$\tilde{A} = \left\{((x,u), \mu_{\tilde{A}}(x,u)) | x \in X, u \in J_X \subset [0,1]\right\}.$$

In particular, we have type-2 fuzzy set $\tilde{A}$:

$$\tilde{A} = \left\{((x, \mu_1(x)), \mu_2(\mu_1(x), u)) | x \in X, u \in [0,1]\right\},$$

where μ_1 is called the primary membership function and μ_2 is called the secondary membership function.

Here, we would extend triangular type-1 fuzzy numbers to type-2. It is very important to define the principal set as a symmetric triangular type-1 fuzzy number. We have call this new type-2 fuzzy numbers "Triangular Shaped Type-2 Fuzzy Numbers".

Definition 2.7. *Triangular shaped type-2 fuzzy numbers.*
A triangular shaped type-2 fuzzy number $\tilde{A}_1$ on $\mathbb{R}$ is defined by the following:

$$\mu_1(x) = \langle\langle a-1, a, a+1\rangle\rangle(x) \max\{1 - |x-a|, 0\},$$
$$\mu_2(\mu_1(x), u) = \begin{cases} 1 - \dfrac{|u-\mu_1|}{\mu_1 \wedge (1-\mu_1)}, & \mu_1 \neq 0, 1 \\ \begin{cases} 1, & u = \mu_1 \\ 0, & otherwise \end{cases}, & \mu_1 = 0, 1 \end{cases}.$$

Example 2.1. *Triangular shaped type-2 fuzzy numbers.*
Let

$$\mu_1(x) = \langle\langle 1, 2, 3\rangle\rangle(x), \quad \mu_2(x, u) = \begin{cases} 1 - \dfrac{|u-\mu_1|}{\mu_1 \wedge (1-\mu_1)}, & \mu_1 \neq 0, 1 \\ \begin{cases} 1, & u = \mu_1 \\ 0, & otherwise \end{cases}, & \mu_1 = 0, 1 \end{cases}.$$

Then, plotting the point $(x, u, \mu_2(x,u))$ in 3D space, we get the surface of the type-2 fuzzy set in Figure 2.

For type-2 fuzzy sets, we could consider α-cut sets as in type-1. For this purpose, we define the β-plane.

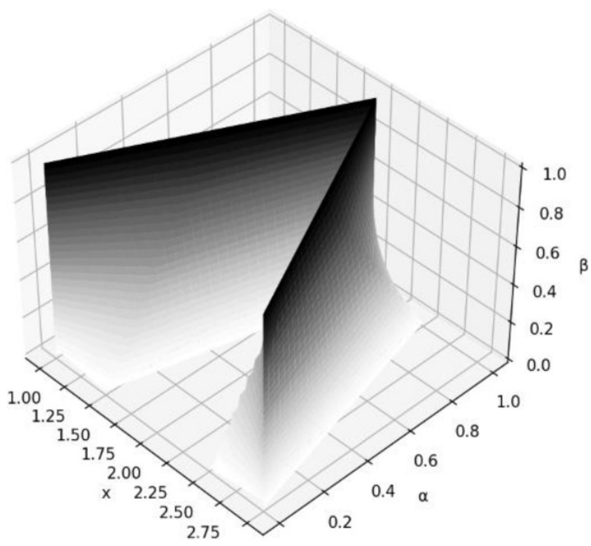

FIGURE 2. Type-2 fuzzy set

Definition 2.8. *β-plane* [10].
Let $\tilde{A} \in \mathcal{F}^2(X)$, β-plane of $\tilde{A}$ is defined by

$$\left[\tilde{A}\right]_\beta = \left\{(x,u)|\mu_{\tilde{A}}(x,u) \geq \beta\right\}, \quad 0 \leq \beta \leq 1.$$

In particular, we have type-2 fuzzy set $\tilde{A}$:

$$\tilde{A} = \left\{\left((x,\mu_1(x)), \mu_2(x,\mu_1(x))\right)|x \in X\right\},$$

where μ_1 is called the primary membership function and μ_2 is called the secondary membership function.

Remark 2.1. *Mendel et al. refer to it as α-plane [7], but in this paper we refer to it as the β-plane for ease of distinction.*

From Definition 2.8, the type-2 fuzzy set $\tilde{A}$ could be described as follows:

$$\tilde{A} = \bigcup_{0\leq\beta\leq1} \beta\left[\tilde{A}\right]_\beta.$$

Then, α-cut of this β-plane could be considered.

Definition 2.9. *α-cut of β-plane* [10].
Let $\tilde{A} \in \mathcal{F}^2(X)$, if β-plane of $\tilde{A}$ is expressed by $\left[\tilde{A}\right]_\beta = \langle \underline{A}_\beta, \overline{A}_\beta\rangle$, LMF $\underline{A}_\beta$ (Lower Membership Function) and UMF $\overline{A}_\beta$ (Upper Membership Function), then α-cut of β-plane is defined by

$$\left[\tilde{A}\right]_\beta^\alpha = \left\langle \left[\underline{A}_\beta\right]_\alpha, \left[\overline{A}_\beta\right]_\alpha\right\rangle.$$

Theorem 2.2. *Decomposition theorem for type-2 fuzzy set* [11].
Let $\tilde{A} \in \mathcal{F}^2(X)$, the following equality holds

$$\tilde{A} = \bigcup_{0\leq\beta\leq1} \beta \bigcup_{0\leq\alpha\leq1} \alpha\left[\tilde{A}\right]_\beta^\alpha.$$

For type-2 fuzzy set $\tilde{A}$ on the universal set X, the fuzzy bound of the primary membership grade is called the footprint (FOU, Footprint of Uncertainty); the FOU of type-2 fuzzy set $\tilde{A}$ is defined as follows.

Definition 2.10. *Footprint* [9].
Let $\tilde{A} \in \mathcal{F}^2(X)$, the union set of all of the primary membership grade is called the footprint of the type-2 fuzzy set $\tilde{A}$ and is defined by the following:

$$FOU\left(\tilde{A}\right) = \int_X J_X.$$

Footprint is the region obtained by projecting the type-2 fuzzy set onto the xu-plane, which is equal to the β-plane $\left[\tilde{A}\right]_\beta$ $(\beta = 0)$.

Furthermore, Principle Set (PS) of type-2 fuzzy set $\tilde{A}$ is defined by the following.

Definition 2.11. *Principle set* [9].
Let $\tilde{A} \in \mathcal{F}^2(X)$, a set whose secondary membership grade is 1 is called the principle set of the type-2 fuzzy set $\tilde{A}$ and is defined as follows:

$$PS\left(\tilde{A}\right) = \left\{(x,u)|x \in X, \mu_{\tilde{A}}(x,u) = 1\right\}.$$

The principle set is the same set as β-plane $\left[\tilde{A}\right]_\beta$ $(\beta = 1)$.

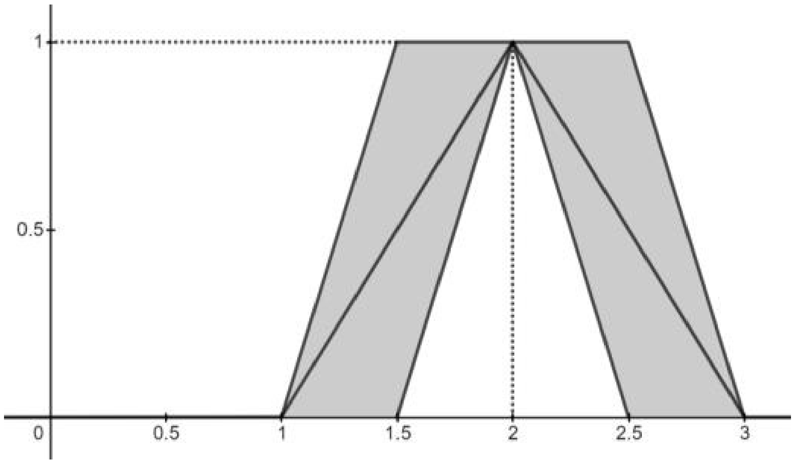

FIGURE 3. Footprint

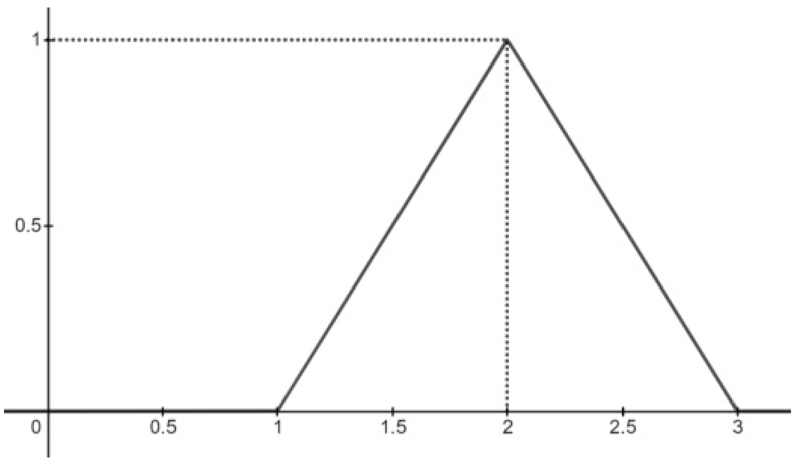

FIGURE 4. Principle set

Example 2.2. *Principle set of Example 2.1.*
For Example 2.1, we could obtain Figure 4.

Next, we consider type-2 fuzzy numbers, which are defined as type-1 fuzzy sets with both normality and convexity. Here, we define a perfect type-2 fuzzy number which is convex and normal.

Definition 2.12. *Perfect type-2 fuzzy numbers* [12].
A type-2 fuzzy set $\tilde{u}$ *on* $\mathbb{R}$ *is called a perfect type-2 fuzzy number if it satisfies the following conditions.*
1) UMF/LMF of $FOU(\tilde{u})$ *are type-1 fuzzy numbers.*
2) UMF/LMF of $PS(\tilde{u})$ *are type-1 fuzzy numbers.*

Example 2.3. *Non perfect type-2 fuzzy numbers.*
If we define the type-2 fuzzy number as follows, the footprint becomes as shown in Figure 5.

$$\mu_1(x) = \langle\langle 1, 2, 3\rangle\rangle(x), \quad \mu_2(x, u) = \max\{1 - 10|u - x|, 0\}\,(0 \leq u \leq 1).$$

From the figure that the lower membership function does not equal 1 clearly, so it is not normal. Therefore, it is not a perfect type-2 fuzzy number.

3. **T-norms.** We would like to consider t-norm operation of type-2 fuzzy numbers. Type-1 fuzzy numbers and type-2 fuzzy numbers are usually computed using the expansion principle, but sometimes they are not fuzzy numbers if disjoint t-norms are used.

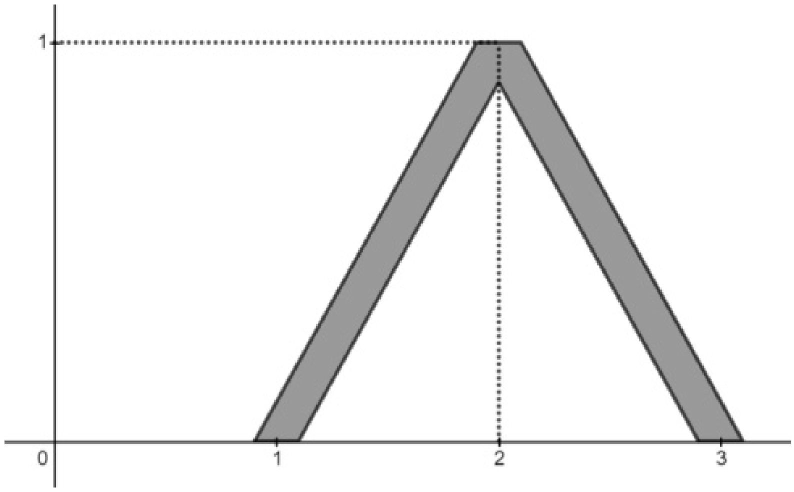

FIGURE 5. Footprint

Definition 3.1. *Uesu product* [6].
Uesu product $T_\lambda(p,q)$ is defined as follows:

$$T_\lambda(p,q) = \begin{cases} p \wedge q, & p \vee q \geq 1-\lambda \\ 0, & \textit{otherwise} \end{cases} .$$

Example 3.1. *Non type-1 fuzzy numbers.*
Consider the type-1 fuzzy numbers 0.2, 0.7*:*

$$\mu_{0.2^*}(x) = \max\left\{0, 1 - \frac{1}{0.2}|x-0.2|\right\}, \quad \mu_{0.7^*}(x) = \max\left\{0, 1 - \frac{1}{0.3}|x-0.7|\right\}.$$

Using Uesu product with $\lambda = 0.3$,

$$T_\lambda(p,q) = \begin{cases} p \wedge q, & p \vee q \geq 1-\lambda \\ 0, & \textit{otherwise} \end{cases} ,$$

then, we have

$$T_\lambda(0.2^*, 0.7^*) = \begin{cases} \mu_{0.2^*}(z), & 0 < z \leq 1 \\ 1, & z = 0 \end{cases} .$$

From Figure 6, this is not convex and is not a type-1 fuzzy number.

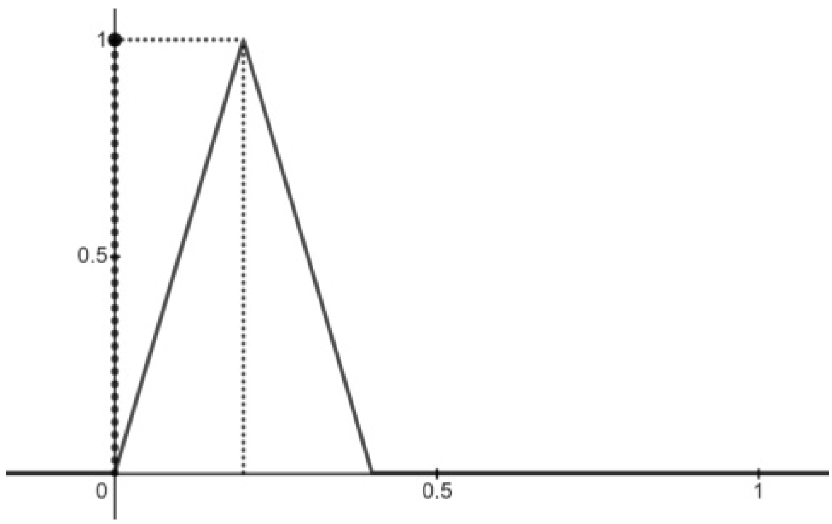

FIGURE 6. Result of Uesu product

Based on this result, we define the operation by t-norm of type-2 fuzzy numbers by α-cut of β-plane.

Definition 3.2. *T-norm operation for type-2 fuzzy numbers.*

Let perfect type-2 fuzzy numbers $\tilde{u}$ and $\tilde{v}$ on $\mathbb{R}$, and their β-planes and α-cut are

$$[\tilde{u}]_\beta^\alpha = \left\langle \left[\underline{u}_\beta\right]_\alpha, \left[\overline{u}_\beta\right]_\alpha \right\rangle = \left\langle \left[\underline{u}_{\beta,-,\alpha}, \underline{u}_{\beta,+,\alpha}\right], \left[\overline{u}_{\beta,-,\alpha}, \overline{u}_{\beta,+,\alpha}\right] \right\rangle,$$
$$[\tilde{v}]_\beta^\alpha = \left\langle \left[\underline{v}_\beta\right]_\alpha, \left[\overline{v}_\beta\right]_\alpha \right\rangle = \left\langle \left[\underline{v}_{\beta,-,\alpha}, \underline{v}_{\beta,+,\alpha}\right], \left[\overline{v}_{\beta,-,\alpha}, \overline{v}_{\beta,+,\alpha}\right] \right\rangle.$$

Then, $T(\tilde{u},\tilde{v})$ is defined by the β-plane and α-cut operations. Here, for simplicity, we omit the subscripts α and β for the endpoints of the α-cut.

$$T(\tilde{u},\tilde{v}) = \bigcup_{0\leq\beta\leq 1} \beta \bigcup_{0\leq\alpha\leq 1} \alpha \left[T(\tilde{u},\tilde{v})\right]_\beta^\alpha,$$

where

$$[T(\tilde{u},\tilde{v})]_\beta^\alpha = \left\langle \left[T\left(\underline{u}_-, \underline{v}_-\right), T\left(\underline{u}_+, \underline{v}_+\right)\right], \left[T\left(\overline{u}_-, \overline{v}_-\right), T\left(\overline{u}_+, \overline{v}_+\right)\right] \right\rangle.$$

Example 3.2. *Non type-1 fuzzy numbers.*

Consider the perfect triangular type-2 fuzzy numbers [12] $\widetilde{0.2}$, $\widetilde{0.7}$:

$$\widetilde{0.2} = \langle\langle 0, 0.05, 0.1; 0.2; 0.3, 0.35, 0.4\rangle\rangle, \quad \widetilde{0.7} = \langle\langle 0.5, 0.55, 0.6; 0.7; 0.8, 0.85, 0.9\rangle\rangle$$

Using Uesu product, we have

$$\left[T_\lambda\left(\widetilde{0.2}, \widetilde{0.7}\right)\right]_\beta^\alpha = \left\langle \left[T_\lambda\left(\underline{u}_-, \underline{v}_-\right), T_\lambda\left(\underline{u}_+, \underline{v}_+\right)\right], \left[T_\lambda\left(\overline{u}_-, \overline{v}_-\right), T_\lambda\left(\overline{u}_+, \overline{v}_+\right)\right] \right\rangle.$$

With $\lambda = 0.5$, we obtain type-2 fuzzy number as shown in Figure 7.

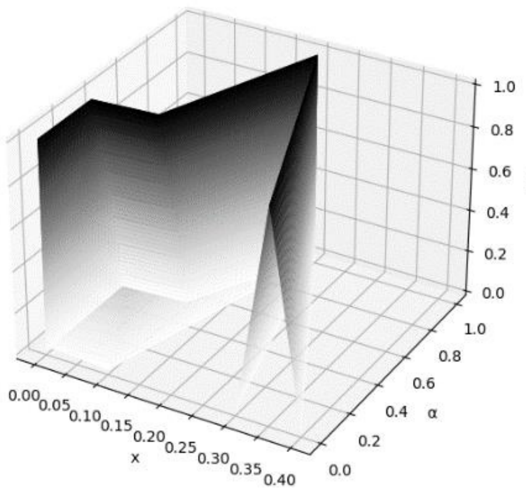

Figure 7. Result of Uesu product

Then, we could also see that the result of this operation is a perfect type-2 fuzzy number.

4. **Conclusions.** By defining the disjoint t-norm operation of type-2 fuzzy numbers as an interval operation, we have generalized the t-norm operation. In the future, we would like to further study type-2 fuzzy numbers and their applications.

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