ICIC Express Letters
Volume **17**, Number **10**, October **2023**

ICIC International ©2023 ISSN 1881-803X
pp. 1095–1102

ON COVERED TWO-SIDED IDEALS OF PARTIALLY ORDERED TERNARY SEMIGROUPS

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Received November 2022; accepted February 2023

Abstract. *In this paper, we introduce the concept of covered two-sided ideals of partially ordered ternary semigroups and study some results based on covered two-sided ideals. A partially ordered ternary semigroup is an algebraic structure under the ternary operator and a partial order satisfying compatible laws. This algebraic structure is generalizations of ordered semigroups and ternary semigroups. We prove that, under some conditions, every proper two-sided ideal of a regular partially ordered ternary semigroup is a covered two-sided ideal.*

Keywords: Partially ordered ternary semigroups, Two-sided ideals, Covered two-sided ideals, Greatest two-sided ideals

1. **Introduction.** Ternary algebraic systems were first studied by Kanser [1], who gave the idea of n-ary algebras. Lehmer [2] studied some ternary algebraic systems called triplex that appear to be commutative ternary groups. Every semigroup can be reduced to a ternary semigroup, but a ternary semigroup generally need not necessarily reduce to a semigroup. The concept of ternary semigroups was known to Banach (cf. [3]) who, by an example, verified that a ternary semigroup does not necessarily reduce to an ordinary semigroup. Later, the concept of ternary semigroups has been educated in various fields [4, 5, 6]. One of generalizations of a ternary semigroup, is so-called a partially ordered ternary semigroup (shortly: po-ternary semigroup). A po-ternary semigroup sometimes was said to be an ordered ternary semigroup. Many mathematicians have studied the properties of po-ternary semigroups. Abbasi et al. [7] focused on studying some generalizations of quasi-ideals. Iampan [8] introduced a po-ternary semigroup and characterized the minimality and maximality of some ideals in po-ternary semigroups. He also introduced and studied the concept of ideal extensions of po-ternary semigroups in [9]. An ideal of some algebraic structures (for example, rings and semigroups) is a special subset of its elements. The

DOI: 10.24507/icicel.17.10.1095

study of ideals in rings and semigroups is one of the important fields of research in ring theory and semigroup theory. Similarly, an ideal theory in other algebraic structures is one of the important research fields.

The inspiration of the target of this research arose from the following. The definition of covered two-sided ideals of semigroups was introduced and studied by Fabrici in [10]. The concept of covered two-sided ideals in semigroups was extended to some generalizations of semigroups, for example, covered two-sided ideals in ordered semigroups [11], and covered two-sided hyperideals in ordered semihypergroups [12]. In [13], Sanborisoot and Changphas introduced the concept of pure ideals in ordered ternary semigroups. In [14], a regular ordered ternary semigroup was studied and described by Pornsurat and Pibaljommee. Recently, Khan et al. [15] introduced and studied the concept of covered lateral ideals of ordered ternary semigroups. Our purpose of this paper is contained in Section 3. In this section, we introduce the concept of covered two-sided ideals of po-ternary semigroups and study some results based on covered two-sided ideals. Finally, in a regular po-ternary semigroup T, we prove that, under some conditions, every proper two-sided ideal of T is a covered two-sided ideal. However, we will recall basic definitions and results in Section 2.

2. **Preliminaries.** For the rest of this section, basic definitions and results used throughout the paper will be recalled.

A *ternary semigroup* is an algebraic structure $(T, [\ ])$ such that T is a non-empty set and $[\ ] : T \times T \times T \to T$ a ternary operation satisfying the associative law, that is,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in T$. For x, y, z in a ternary semigroup T, we will write xyz instead of $[xyz]$.

A ternary semigroup T is said to be a *partially ordered ternary semigroup* (shortly: po-ternary semigroup) if there exists a partially ordered relation $\leq$ such that for any $a, b, x, y \in T$,

$$a \leq b \Rightarrow axy \leq bxy,\ \ xay \leq xby \text{ and } xya \leq xyb.$$

A partially ordered ternary semigroup is also referred to as a po-ternary semigroup or an ordered ternary semigroup. In this paper, we write T for a po-ternary semigroup, unless otherwise specified.

Let A, B, C be non-empty subsets of a po-ternary semigroup T. We denote

$$ABC := \{abc \in T : a \in A, b \in B, c \in C\} \text{ and } (A] := \{t \in T : t \leq a \text{ for some } a \in A\}.$$

Note that for any $A, B, C \subseteq T$, $A \subseteq (A]$, $((A]] = (A]$, $(A](B](C] \subseteq (ABC]$, $(A] \cup (B] = (A \cup B]$ and $A \subseteq B$ implies $(A] \subseteq (B]$.

Let T be a po-ternary semigroup. A non-empty subset A of T is called a *left* (*resp. right, lateral*) *ideal* of T if 1) $TTA \subseteq A$ (resp. $ATT \subseteq A$, $TAT \subseteq A$) and 2) if $a \in A$ and $b \in T$ such that $b \leq a$, then $b \in A$, that is, $(A] = A$. If A is a left and a right ideal of T, then it is called a *two-sided ideal* of T. If A is a left, a right and a lateral ideal of T, then it is called an *ideal* of T. A two-sided ideal A of T is said to be *proper* if $A \neq T$. A proper two-sided ideal A of T is said to be *maximal* if for any two-sided ideal B of T such that $A \subseteq B \subseteq T$, then $A = B$ or $B = T$. If T has no proper two-sided ideals, then it is *two-sided simple*. Moreover, in [14], T is called *regular* if and only if for each $a \in T$ there exist $x, y \in T$ such that $a \leq axaya$.

It is known that the union of two two-sided ideals of T is a two-sided ideal of T, and the intersection of two two-sided ideals of T is a two-sided ideal of T, if it is non-empty. If A is a non-empty subset of T, then the intersection of all two-sided ideals of T containing A, denoted by $(A)_t$, is the *smallest two-sided ideal* of T containing A, and it is of the form [13]

$$(A)_t = (A \cup TTA \cup ATT \cup TTATT].$$

For $a \in T$, we write $(a)_t$ simply for $(\{a\})_t$, and called the *principal two-sided ideal* of T generated by $a \in T$. Further, we have

$$(a)_t = (a \cup TTa \cup aTT \cup TTaTT].$$

3. **Main Results.**

Lemma 3.1. *Let A be a non-empty subset of a po-ternary semigroup T. Then the following statements hold:*

1) $(TTATT]$ is a two-sided ideal of T;

2) $(TTA \cup ATT \cup TTATT]$ is a two-sided ideal of T;

3) for any $a \in T$, if $a \in (TTATT]$, then $(a)_t \subseteq (TTATT]$.

Proof: 1) Let $B = (TTATT]$. We consider

$$\begin{aligned} TTB &= TT(TTATT] \\ &\subseteq (T](T](TTATT] \\ &\subseteq (TT(TTATT)] \\ &= ((TTT)TATT] \\ &\subseteq (TTATT] \subseteq B, \end{aligned}$$

$$\begin{aligned} BTT &= (TTATT]TT \\ &\subseteq (TTATT](T](T] \\ &\subseteq ((TTATT)TT] \\ &= (TTAT(TTT)] \\ &\subseteq (TTATT] \subseteq B \end{aligned}$$

and $(B] = ((TTATT]] = (TTATT] = B$. Thus, B is a two-sided ideal of T.

2) It is similar to 1).

3) Let $a \in T$. Suppose that $a \in (TTATT]$. Since $(TTATT]$ is a two-sided ideal of T, then

$$\begin{aligned} &TTa \subseteq TT(TTATT] \subseteq (TTATT], \\ &aTT \subseteq (TTATT]TT \subseteq (TTATT] \text{ and} \\ &TTaTT \subseteq TT(TTATT]TT \subseteq (TTATT]. \end{aligned}$$

Thus, $a \cup TTa \cup aTT \cup TTaTT \subseteq (TTATT]$, and so $(a)_t = (a \cup TTa \cup aTT \cup TTaTT] \subseteq ((TTATT]] = (TTATT]$. Hence, $(a)_t \subseteq (TTATT]$. $\square$

Definition 3.1. *Any proper two-sided ideal A of a po-ternary semigroup T is called a* covered two-sided ideal *(CT-ideal) of T if*

$$A \subseteq (TT(T - A) \cup (T - A)TT \cup TT(T - A)TT].$$

Example 3.1. *Let $T = \{0, 1, 2, 3, 4\}$ and $xyz = x * (y * z)$ for all $x, y, z \in T$, where $*$ is defined by the table*

$*$	0	1	2	3	4
0	0	0	2	3	0
1	0	1	2	3	0
2	0	0	2	3	0
3	0	0	2	3	0
4	0	0	2	3	4

and the partial order defined by

$$\leq := \{(0,0),(0,1),(0,2),(0,3),(0,4),(1,1),(2,2),(2,3),(3,3),(4,4)\}.$$

We can check that T is a po-ternary semigroup with proper two-sided ideals: $\{0,2,3\}$, $\{0,1,2,3\}$ and $\{0,2,3,4\}$. The proper two-sided ideal $\{0,2,3\}$ is a CT-ideal of T, but the proper two-sided ideals $\{0,1,2,3\}$ and $\{0,2,3,4\}$ are not CT-ideals of T.

Theorem 3.1. *Let A and B be any two different proper two-sided ideals of a po-ternary semigroup T. If $A \cup B = T$, then both A and B are not CT-ideals of T.*

Proof: Assume that $A \cup B = T$. Suppose that A is a CT-ideal of T. We have $A \subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT]$. Since $A \cup B = T$, then $T - A \subseteq B$. Thus,

$$\begin{aligned} A &\subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT] \\ &\subseteq (TTB \cup BTT \cup (TTB)TT] \\ &\subseteq (B \cup B \cup BTT] \\ &\subseteq (B] = B. \end{aligned}$$

Since $A \subseteq B$ and $A \cup B = T$, we obtain that $T = B$. This is a contradiction. Hence, A is not a CT-ideal of T. Similarly, if B is a CT-ideal of T, then $T = A$ which is a contradiction. Hence, B is not a CT-ideal of T. □

Corollary 3.1. *If a po-ternary semigroup T contains more than one maximal proper two-sided ideal, then all maximal proper two-sided ideals are not CT-ideals of T.*

Proof: Assume that T contains two maximal different proper two-sided ideals A and B. It is well-known that the union of two two-sided ideals is a two-sided ideal. Then $A \cup B$ is a two-sided ideal of T. Since $A \subset A \cup B$ and A is a maximal proper two-sided ideal of T, it follows that $A \cup B = T$. Hence, by Theorem 3.1, we conclude that A and B are not CT-ideals of T. □

Theorem 3.2. *Let A and B be any two CT-ideals of a po-ternary semigroup T. If $A \cup B \neq T$, then $A \cup B$ is a CT-ideal of T.*

Proof: Assume that $A \cup B \neq T$. Then $A \cup B$ is a proper two-sided ideal of T. Now, to show that $A \cup B$ is a CT-ideal of T. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ and A is a CT-ideal of T, we have

$$x \in A \subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT].$$

Then there exist $a_1, a_2, a_3 \in T - A$ such that $x \in (TTa_1 \cup a_2TT \cup TTa_3TT]$. There are eight cases to consider:

Case 1: $a_1, a_2, a_3 \in (T-A) - B$. Then $a_1, a_2, a_3 \in T - (A \cup B)$, and so

$$\begin{aligned} x &\in (TTa_1 \cup a_2TT \cup TTa_3TT] \\ &\subseteq (TT(T-(A \cup B)) \cup (T-(A \cup B))TT \cup TT(T-(A \cup B))TT]. \end{aligned}$$

Case 2: $a_1, a_2, a_3 \in (T-A) \cap B$. Then $a_1, a_2, a_3 \in (T-A)$ and $a_1, a_2, a_3 \in B$. Thus,

$$a_1, a_2, a_3 \in B \subseteq (TT(T-B) \cup (T-B)TT \cup TT(T-B)TT].$$

Hence, there exist $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9 \in T - B$ such that $a_1 \in (TTb_1 \cup b_2TT \cup TTb_3TT]$, $a_2 \in (TTb_4 \cup b_5TT \cup TTb_6TT]$ and $a_3 \in (TTb_7 \cup b_8TT \cup TTb_9TT]$. If $b_1, b_2, b_3 \in A$, then $a_1 \in (TTb_1 \cup b_2TT \cup TTb_3TT] \subseteq (TTA \cup ATT \cup TTATT] \subseteq (A] = A$, and so $a_1 \in A$. This is a contradiction. Similarly, if $b_4, b_5, b_6, b_7, b_8, b_9 \in A$, it follows that $a_2, a_3 \in A$ which is a contradiction. Thus, $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9 \in T - A$, and so $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9 \in (T-A) \cap (T-B) = T - (A \cup B)$. Hence,

$$\begin{aligned} x &\in (TTa_1 \cup a_2TT \cup TTa_3TT] \\ &\subseteq ((T](T](TTb_1 \cup b_2TT \cup TTb_3TT] \cup (TTb_4 \cup b_5TT \cup TTb_6TT](T](T] \\ &\quad \cup (T](T](TTb_7 \cup b_8TT \cup TTb_9TT](T](T]] \\ &\subseteq (((TTT)Tb_1 \cup TTb_2TT \cup (TTT)Tb_3TT] \cup (TTb_4TT \cup b_5T(TTT) \end{aligned}$$

$$\begin{aligned}
&\cup TTb_6T(TTT)] \cup ((TTT)Tb_7TT \cup TTb_8T(TTT) \cup (TTT)Tb_9T(TTT)]] \\
&\subseteq ((TTb_1 \cup TTb_2TT \cup TTb_3TT \cup TTb_4TT \cup b_5TT \cup TTb_6TT \\
&\quad \cup TTb_7TT \cup TTb_8TT \cup TTb_9TT]] \\
&= (TTb_1 \cup TTb_2TT \cup TTb_3TT \cup TTb_4TT \cup b_5TT \cup TTb_6TT \cup TTb_7TT \\
&\quad \cup TTb_8TT \cup TTb_9TT] \\
&\subseteq (TT(T-(A\cup B)) \cup TT(T-(A\cup B))TT \cup TT(T-(A\cup B))TT \\
&\quad \cup TT(T-(A\cup B))TT \cup (T-(A\cup B))TT \cup TT(T-(A\cup B))TT \\
&\quad \cup TT(T-(A\cup B))TT \cup TT(T-(A\cup B))TT \cup TT(T-(A\cup B))TT] \\
&= (TT(T-(A\cup B)) \cup (T-(A\cup B))TT \cup TT(T-(A\cup B))TT].
\end{aligned}$$

Case 3: $a_1, a_2 \in T-(A\cup B)$ and $a_3 \in (T-A)\cap B$. Since $a_3 \in B$ and B is a CT-ideal of T, we have $a_3 \in (TTc_1 \cup c_2TT \cup TTc_3TT]$ for some $c_1, c_2, c_3 \in T-B$. Then we can proceed similar to Case 2 and we obtain $c_1, c_2, c_3 \in T-(A\cup B)$. Hence,

$$\begin{aligned}
x &\in (TTa_1 \cup a_2TT \cup TTa_3TT] \\
&\subseteq (TTa_1 \cup a_2TT \cup (T](T](TTc_1 \cup c_2TT \cup TTc_3TT](T](T]] \\
&\subseteq (TTa_1 \cup a_2TT \cup ((TTT)Tc_1TT \cup TTc_2T(TTT) \cup (TTT)Tc_3T(TTT)]] \\
&\subseteq (TTa_1 \cup a_2TT \cup TTc_1TT \cup TTc_2TT \cup TTc_3TT] \\
&\subseteq (TT(T-(A\cup B)) \cup (T-(A\cup B))TT \cup TT(T-(A\cup B))TT \\
&\quad \cup TT(T-(A\cup B))TT \cup TT(T-(A\cup B))TT] \\
&= (TT(T-(A\cup B)) \cup (T-(A\cup B))TT \cup TT(T-(A\cup B))TT].
\end{aligned}$$

Case 4: $a_1, a_3 \in T-(A\cup B)$ and $a_2 \in (T-A)\cap B$. Then this is similar to Case 3.
Case 5: $a_2, a_3 \in T-(A\cup B)$ and $a_1 \in (T-A)\cap B$. Then this is similar to Case 3.
Case 6: $a_1 \in T-(A\cup B)$ and $a_2, a_3 \in (T-A)\cap B$. Then this is similar to Case 3.
Case 7: $a_2 \in T-(A\cup B)$ and $a_1, a_3 \in (T-A)\cap B$. Then this is similar to Case 3.
Case 8: $a_3 \in T-(A\cup B)$ and $a_1, a_2 \in (T-A)\cap B$. Then this is similar to Case 3.

In all these cases, we infer that

$$x \in (TT(T-(A\cup B)) \cup (T-(A\cup B))TT \cup TT(T-(A\cup B))TT].$$

Similarly, we can prove this for $x \in B$. Thus we conclude that

$$A\cup B \subseteq (TT(T-(A\cup B)) \cup (T-(A\cup B))TT \cup TT(T-(A\cup B))TT].$$

This shows that $A\cup B$ is a CT-ideal of T. $\square$

Theorem 3.3. *Let T be a po-ternary semigroup. If A is a two-sided ideal and B is a CT-ideal of T such that $A\cap B \neq \varnothing$, then $A\cap B$ is a CT-ideal of T.*

Proof: Assume that A is a two-sided ideal and B is a CT-ideal of T such that $A\cap B \neq \varnothing$. We have $A\cap B$ is a proper two-sided ideal of T and $A\cap B \subseteq B$. Then $T-B \subseteq T-(A\cap B)$. Thus, it follows by assumption that

$$\begin{aligned}
A\cap B \subseteq B &\subseteq (TT(T-B) \cup (T-B)TT \cup TT(T-B)TT] \\
&\subseteq (TT(T-(A\cap B)) \cup (T-(A\cap B))TT \cup TT(T-(A\cap B))TT].
\end{aligned}$$

This shows that $A\cap B$ is a CT-ideal of T. $\square$

The following corollary is an immediate consequence of Theorem 3.3 since every CT-ideal of a po-ternary semigroup T is a two-sided ideal of T.

Corollary 3.2. *If A and B are two CT-ideals of a po-ternary semigroup T such that $A\cap B \neq \varnothing$, then $A\cap B$ is a CT-ideal of T.*

Theorem 3.4. *Let A be any two-sided ideal of a po-ternary semigroup T. If $A \subset (TTt \cup tTT \cup TTtTT]$ and $A \neq (TTt \cup tTT \cup TTtTT]$ for some $t \in T$, then A is a CT-ideal of T.*

Proof: Assume that $A \subset (TTt \cup tTT \cup TTtTT]$ and $A \neq (TTt \cup tTT \cup TTtTT]$ for some $t \in T$. Then A is a proper two-sided ideal of T. Now, we claim that $t \in T - A$. Suppose that $t \in A$. We have

$$A \subset (TTt \cup tTT \cup TTtTT] \subseteq (TTA \cup ATT \cup TTATT] \subseteq (A] = A.$$

So, we obtain $A = (TTt \cup tTT \cup TTtTT]$. This is a contradiction. Thus, $t \in T - A$. Hence, $A \subset (TTt \cup tTT \cup TTtTT] \subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT]$. This shows that A is a CT-ideal of T. □

Corollary 3.3. *A po-ternary semigroup T in which an element t does not belong to $(TTt \cup tTT \cup TTtTT]$ contains a CT-ideal.*

Proof: Suppose that $A = (TTt \cup tTT \cup TTtTT]$. We have A is a two-sided ideal of T. If $t \notin A$, then A is a proper two-sided ideal of T, and so

$$A = (TTt \cup tTT \cup TTtTT] \subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT].$$

Hence, A is a CT-ideal of T. □

Theorem 3.5. *Let T be a po-ternary semigroup. If T is not two-sided simple such that there is no any two proper two-sided ideals in which their intersection is empty, then T contains at least one CT-ideal.*

Proof: Assume that T is not two-sided simple such that there are no any two proper two-sided ideals in which their intersection is empty. Let A be a proper two-sided ideal of T. Now, we set $B = (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT]$. Then B is a two-sided ideal of T. By assumption, $A \cap B \neq \varnothing$, and so $A \cap B$ is a proper two-sided ideal of T. Suppose that $C = A \cap B$. Then $C \subseteq A$ and $C \subseteq B$. Since $C \subseteq A$, we have $T - A \subseteq T - C$. Since $C \subseteq B$, it follows that

$$\begin{aligned} C \subseteq B &= (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT] \\ &\subseteq (TT(T-C) \cup (T-C)TT \cup TT(T-C)TT]. \end{aligned}$$

This shows that C is a CT-ideal of T. □

Let T be a po-ternary semigroup. A proper two-sided ideal I^* of T is called the *greatest two-sided ideal* of T if it contains every proper two-sided ideal of T.

Theorem 3.6. *Let T be a po-ternary semigroup. If T contains the greatest two-sided ideal I^* which is a CT-ideal of T, then every proper two-sided ideal of T is a CT-ideal of T.*

Proof: Assume that the greatest two-sided ideal I^* is a CT-ideal of T. Suppose that A is a proper two-sided ideal of T. Then $A \subseteq I^*$, and so $T - I^* \subseteq T - A$. Thus, it follows by assumption that

$$\begin{aligned} A \subseteq I^* &\subseteq (TT(T-I^*) \cup (T-I^*)TT \cup TT(T-I^*)TT] \\ &\subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT]. \end{aligned}$$

This shows that A is a CT-ideal of T. □

Example 3.2. Let $T = \{0, 1, 2, 3, 4\}$ and $xyz = x * (y * z)$ for all $x, y, z \in T$, where $*$ is defined by the table

$*$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	2	2	2
3	0	0	2	3	3
4	0	1	2	3	4

and the partial order defined by

$$\leq := \{(0,0),(0,1),(0,2),(0,3),(0,4),(1,1),(2,2),(3,3),(4,4)\}.$$

We can check that T is a po-ternary semigroup with proper two-sided ideals: $\{0\}$, $\{0,1\}$, $\{0,2\}$, $\{0,1,2\}$ and $\{0,1,2,3\}$. We have the greatest two-sided ideal $I^* = \{0,1,2,3\}$ and I^* is a CT-ideal of T. By Theorem 3.6, we conclude that the proper two-sided ideals $\{0\}$, $\{0,1\}$, $\{0,2\}$, $\{0,1,2\}$ and $\{0,1,2,3\}$ are CT-ideals of T.

Theorem 3.7. *Let A be any proper two-sided ideal of a regular po-ternary semigroup T. If for every principal two-sided ideal $(a)_t \subseteq A$, there exists $b \in T - A$ such that $(a)_t \subseteq (b)_t$, then every proper two-sided ideal of T is a CT-ideal of T.*

Proof: First, we claim that $T = (T^5]$. Clearly, $(T^5] \subseteq T$. Let $x \in T$. Since T is regular, then there exist $y, z \in T$ such that $x \leq xyxzx$. Since $x \leq xyxzx$ and $xyxzx \in TTTTT$, we obtain that $x \in (T^5]$ and so $T \subseteq (T^5]$. Thus, $T = (T^5]$. Next, suppose that for any two-sided ideal A of T and $a \in A$ such that $(a)_t \subseteq A$, there exists $b \in T - A$ such that $(a)_t \subseteq (b)_t$. Since $b \in T$ and $T = (T^5]$, we have $b \in (TTtTT]$ for some $t \in T$. If $t \in A$, it follows that $b \in (TTATT] \subseteq (A] = A$ which is a contradiction. Thus, $t \in T - A$. Hence, $b \in (TT(T-A)TT]$, and by Lemma 3.1, $(b)_t \subseteq (TT(T-A)TT]$. Since $(a)_t \subseteq (b)_t$, it follows that

$$a \in (a)_t \subseteq (b)_t \subseteq (TT(T-A)TT] \subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT].$$

Thus, $A \subseteq (TT(T-A) \cup (T-A)TT \cup TT(T-A)TT]$. This shows that A is a CT-ideal of T. $\square$

4. **Conclusions.** In this paper, we introduce the concept of covered two-sided ideals (shortly, CT-ideals) of po-ternary semigroups. The union of two CT-ideals is also a CT-ideal if this union is proper (Theorem 3.2) and the intersection of two CT-ideals of a po-ternary semigroup is also a CT-ideal if this intersection is nonempty (Theorem 3.3). Moreover, we investigate significant results for CT-ideals in Theorems 3.4, 3.5, 3.6 and 3.7.

In the future, we can extend some results of partially ordered semigroups and/or ternary semigroups to the results of partially ordered ternary semigroups.

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