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INTUITIONISTIC FUZZY COMPARATIVE UP-FILTERS AND THEIR LEVEL SUBSETS

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Abstract. *In this paper, we introduce the concept of intuitionistic fuzzy comparative UP-filters and investigate their properties. Also, we discuss the relationship between intuitionistic fuzzy comparative UP-filters and fuzzy comparative UP-filters. Moreover, we establish the concept of complement and level subset with the intuitionistic fuzzy comparative UP-filters.*
Keywords: Comparative UP-filters, UP-filters, Intuitionistic fuzzy UP-filters, Intuitionistic fuzzy comparative UP-filters

1. **Introduction.** In 2017, Iampan [1] introduced the concept of UP-algebras as a generalization of KU-algebras [2]. UP-algebra is an algebraic structure type of logic from an introductory algebra class. Many researchers brought this concept of UP-algebra into various concepts, such as UP-algebra with fuzzy sets [3], picture fuzzy sets [4], bipolar fuzzy sets [5], neutrosophic sets [6], and intuitionistic fuzzy sets [7].

The expansion of the UP-algebra concept to a new notion has been attractive to various researchers. For example, Jun and Iampan [8] introduced the concept of comparative

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and allied UP-filters and investigated several properties. In 2019, they discussed the relationship between a UP-filter and a comparative UP-filter, conditions for a UP-filter to be a comparative UP-filter, and characterizations of a (comparative) UP-filter. In 2022, Gaketem et al. [9] proposed the concept of bipolar fuzzy comparative UP-filters, investigated their properties, and expressed bipolar fuzzy comparative UP-filters to neutrosophic sets. Expanding this concept into different sets, like the algebraic structure is essential. In 1986, Atanassov [10] studied the concept of intuitionistic fuzzy sets as a generalization of the concept of fuzzy sets of Zadeh [11]. The concept of fuzzy sets that expresses uncertainties is an important mathematical tool for solving theoretical problems. In 2019, Thongngam and Iampan [12] studied the concept of intuitionistic fuzzy UP-filters and intuitionistic fuzzy near UP-filters. In 2020, Abdullah and Shadhan [13] applied the concept of intuitionistic fuzzy sets on Q-algebras. In addition, Songsaeng et al. [14] have also studied neutrosophic comparative UP-filters of UP-algebras in 2021. In 2023, Khamrot et al. [15] introduced the concept of intuitionistic fuzzy implicative UP-filters of UP-algebras and provided some properties of intuitionistic fuzzy implicative UP-filters and together studied the relation of intuitionistic fuzzy implicative UP-filters and intuitionistic fuzzy UP-filters in UP-algebras.

We are interested in extending the notion of comparative UP-filters to intuitionistic fuzzy comparative UP-filters (IFCUPFs) to supplement the intuitionistic fuzzy set notion of UP-algebras. This article aims to introduce the new concept of IFCUPFs in detail below and give some definitions, properties, and examples of UP-algebras in the next section. As a result, we find a relationship between IFCUPFs and their level subsets and complements. Finally, we conclude and plan to future work.

2. Preliminaries.

Definition 2.1. [1] *An algebra* $\left(\tilde{A}, \star, 0\right)$ *of type* $(2,0)$ *is called a UP-algebra, where* $\tilde{A}$ *is a nonempty set,* $\star$ *is a binary operation on* $\tilde{A}$*, and* 0 *is a fixed element of* $\tilde{A}$ *(i.e., a nullary operation) if it satisfies the following axioms: (i)* $\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right)((\tilde{y} \star \tilde{z}) \star ((\tilde{x} \star \tilde{y}) \star (\tilde{x} \star \tilde{z})) = 0)$*, (ii)* $\left(\text{for all } \tilde{x} \in \tilde{A}\right)(0 \star \tilde{x} = \tilde{x})$*, (iii)* $\left(\text{for all } \tilde{x} \in \tilde{A}\right)(p \star 0 = 0)$*, and (iv)* $\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right)(\tilde{x} \star \tilde{y} = 0,\ \tilde{y} \star \tilde{x} = 0 \Rightarrow \tilde{x} = \tilde{y})$*.*

Unless otherwise indicated, we will assume that $\tilde{A}$ is a UP-algebra $\left(\tilde{A}, \star, 0\right)$.

Proposition 2.1. [1] *In a UP-algebra* $\tilde{A}$*, the following properties hold: (i)* $\left(\text{for all } \tilde{x} \in \tilde{A}\right)(\tilde{x} \star \tilde{x} = 0)$*, (ii)* $\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right)(\tilde{x} \star \tilde{y} = 0,\ \tilde{y} \star \tilde{z} = 0 \Rightarrow \tilde{x} \star \tilde{z} = 0)$*, (iii)* $\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right)(\tilde{x} \star \tilde{y} = 0 \Rightarrow (\tilde{z} \star \tilde{x}) \star (\tilde{z} \star \tilde{y}) = 0)$*, (iv)* $\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right)(\tilde{x} \star \tilde{y} = 0 \Rightarrow (\tilde{y} \star \tilde{z}) \star (\tilde{x} \star \tilde{z}) = 0)$*, (v)* $\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right)(\tilde{x} \star (\tilde{y} \star \tilde{x}) = 0)$*, (vi)* $\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right)((\tilde{y} \star \tilde{x}) \star \tilde{x} = 0 \Leftrightarrow \tilde{x} = \tilde{y} \star \tilde{x})$*, and (vii)* $\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right)(\tilde{x} \star (\tilde{y} \star \tilde{y}) = 0)$*.*

For examples of UP-algebras, there have been several interesting research studies (see [16, 17, 18, 19, 20]).

The binary relation $\leq$ on a UP-algebra $\tilde{A}$ is defined as follows: $\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right)(\tilde{x} \leq \tilde{y} \Leftrightarrow \tilde{x} \star \tilde{y} = 0)$ and the following assertions are valid (see [1, 17]).

Next, we recall the concepts of UP-subalgebras, UP-ideals, UP-filters, comparative UP-filters, and implicative UP-filters of UP-algebras [1, 21] as the following definition.

Definition 2.2. *A nonempty subset $\tilde{Z}$ of $\tilde{A}$ is called*
(i) a UP-subalgebra (UPS) of $\tilde{A}$ if

$$\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{Z}\right) \left(\tilde{x} \star \tilde{y} \in \tilde{Z}\right), \tag{1}$$

(ii) a UP-ideal (UPI) of $\tilde{A}$ if

$$0 \in \tilde{Z}, \tag{2}$$

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) \left(\tilde{x} \star (\tilde{y} \star \tilde{z}) \in \tilde{Z},\ \tilde{y} \in \tilde{Z} \Rightarrow \tilde{x} \star \tilde{z} \in \tilde{Z}\right), \tag{3}$$

(iii) a UP-filter (UPF) of $\tilde{A}$ if it satisfies (2) *and*

$$\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right) \left(\tilde{x} \in \tilde{Z},\ \tilde{x} \star \tilde{y} \in \tilde{Z} \Rightarrow \tilde{y} \in \tilde{Z}\right), \tag{4}$$

(iv) a comparative UP-filter (CUPF) of $\tilde{A}$ if it satisfies (2) *and*

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) \left(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y}) \in \tilde{Z},\ \tilde{x} \in \tilde{Z} \Rightarrow \tilde{y} \in \tilde{Z}\right). \tag{5}$$

Clearly, $\tilde{A}$ and $\{0\}$ are UPSs and UPIs of $\tilde{A}$. Every CUPF is a UPF, but the converse is not generally valid, as shown in Jun and Iampan [8].

Theorem 2.1. [16] *Let $\mathfrak{K}$ be a nonempty family of UPSs (resp., UPFs, UPIs, CUPFs) of $\tilde{A}$. Then $\cap\mathfrak{K}$ is a UPS (resp., UPF, UPI, CUPF) of $\tilde{A}$.*

Definition 2.3. *A fuzzy set (FS) ω in a nonempty set $\tilde{Z}$ is a function from $\tilde{Z}$ into the unit closed interval* $[0, 1]$ *of real numbers, i.e., $\omega : \tilde{Z} \to [0, 1]$.*

For any two FSs ω_1 and ω_2 in a nonempty set $\tilde{Z}$, we define (i) $\omega_1 \geq \omega_2 \Leftrightarrow \omega_1(\tilde{x}) \geq \omega_2(\tilde{x})$ for all $\tilde{x} \in \tilde{Z}$, (ii) $\omega_1 = \omega_2 \Leftrightarrow \omega_1 \geq \omega_2$ and $\omega_2 \geq \omega_1$, and (iii) $(\omega_1 \wedge \omega_2)(\tilde{x}) = \min\{\omega_1(\tilde{x}), \omega_2(\tilde{x})\}$ for all $\tilde{x} \in \tilde{Z}$.

Definition 2.4. *Let ω be an FS in $\tilde{Z}$. The FS $\overline{\omega}$ is defined by $\overline{\omega}(\tilde{x}) = 1 - \omega(\tilde{x})$ for all $\tilde{x} \in \tilde{Z}$. We called $\overline{\omega}$ a* complement *of ω in $\tilde{Z}$.*

Definition 2.5. *An FS ω of a UP-algebra $\tilde{A}$ is called*
(i) a fuzzy UP-subalgebra (FUPS) of $\tilde{A}$ if

$$\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right) (\omega(\tilde{x} \star \tilde{y}) \geq \min\{\omega(\tilde{x}), \omega(\tilde{y})\}), \tag{6}$$

(ii) a fuzzy UP-ideal (FUPI) of $\tilde{A}$ if

$$\left(\text{for all } \tilde{x} \in \tilde{A}\right) (\omega(0) \geq \omega(\tilde{x})), \tag{7}$$

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) (\omega(\tilde{x} \star \tilde{z}) \geq \min\{\omega(\tilde{x} \star (\tilde{y} \star \tilde{z})), \omega(\tilde{y})\}), \tag{8}$$

(iii) a fuzzy UP-filter (FUPF) of $\tilde{A}$ if it satisfies (7) *and*

$$\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right) (\omega(\tilde{y}) \geq \min\{\omega(\tilde{x}), \omega(\tilde{x} \star \tilde{y})\}), \tag{9}$$

(iv) a fuzzy comparative UP-filter (FCUPF) of $\tilde{A}$ if it satisfies (7) *and*

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) (\omega(\tilde{y}) \geq \min\{\omega(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \omega(\tilde{x})\}). \tag{10}$$

It easily proves that if ω_1 and ω_2 are FUPS (resp., FUPI) of a UP-algebra $\tilde{A}$, then $\omega_1 \wedge \omega_2$ is also an FUPS (resp., FUPI) of $\tilde{A}$.

Definition 2.6. *Let $\tilde{Z}$ be the universe set. An intuitionistic fuzzy set (IFS) in $\tilde{Z}$ is an object having the form $F := \left\{(\tilde{x}, \omega_F(\tilde{x}), \delta_F(\tilde{x}))\middle| \tilde{x} \in \tilde{Z}\right\}$, where $\omega_F : \tilde{Z} \to [0,1]$ and $\delta_F : \tilde{Z} \to [0,1]$ denote the degree of membership and degree of nonmembership, respectively, and for all $\tilde{x} \in \tilde{Z}$, $0 \leq \omega_F(\tilde{x}) + \delta_F(\tilde{x}) \leq 1$.*

We shall use the symbol $F = (\omega_F, \delta_F)$ for the IFS $F = \left\{(\tilde{x}, \omega_F(\tilde{x}), \delta_F(\tilde{x}))\middle| \tilde{x} \in \tilde{Z}\right\}$ for the sake of notational simplicity.

Kesorn et al. [7] and Thongngam and Iampan [12] introduced the concepts of intuitionistic fuzzy UP-subalgebras, intuitionistic fuzzy UP-ideals, and intuitionistic fuzzy UP-filters of UP-algebras as follows.

Definition 2.7. [7] *An IFS $F = (\omega_F, \delta_F)$ in $\tilde{A}$ is called an intuitionistic fuzzy UP-subalgebra (IFUPS) of $\tilde{A}$ if it satisfies the following conditions: $\omega_F(\tilde{x} \star \tilde{y}) \geq \min\{\omega_F(\tilde{x}), \omega_F(\tilde{y})\}$ and $\delta_F(\tilde{x} \star \tilde{y}) \leq \max\{\delta_F(\tilde{x}), \delta_F(\tilde{y})\}$ for all $\tilde{x}, \tilde{y} \in \tilde{A}$.*

Definition 2.8. [7] *An IFS $F = (\omega_F, \delta_F)$ in $\tilde{A}$ is called an intuitionistic fuzzy UP-ideal (IFUPI) of $\tilde{A}$ if*

$$\left(\text{for all } \tilde{x} \in \tilde{A}\right) (\omega_F(0) \geq \omega_F(\tilde{x})), \tag{11}$$

$$\left(\text{for all } \tilde{x} \in \tilde{A}\right) (\delta_F(0) \leq \delta_F(\tilde{x})), \tag{12}$$

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) (\omega_F(\tilde{x} \star \tilde{z}) \geq \min\{\omega_F(\tilde{x} \star (\tilde{y} \star \tilde{z})), \omega_F(\tilde{y})\}), \tag{13}$$

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) (\delta_F(\tilde{x} \star \tilde{z}) \leq \max\{\delta_F(\tilde{x} \star (\tilde{y} \star \tilde{z})), \delta_F(\tilde{y})\}). \tag{14}$$

Definition 2.9. [12] *An IFS $F = (\omega_F, \delta_F)$ in a UP-algebra $\tilde{A}$ is called an intuitionistic fuzzy UP-filter (IFUPF) of $\tilde{A}$ if it satisfies* (11), (12), *and*

$$\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right) (\omega_F(\tilde{y}) \geq \min\{\omega_F(\tilde{x} \star \tilde{y}), \omega_F(\tilde{x})\}), \tag{15}$$

$$\left(\text{for all } \tilde{x}, \tilde{y} \in \tilde{A}\right) (\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x} \star \tilde{y}), \delta_F(\tilde{x})\}). \tag{16}$$

3. Intuitionistic Fuzzy Comparative UP-Filters.

This section shows the main results. We introduce IFCUPFss and investigate their properties.

Definition 3.1. *An IFS $F = (\omega_F, \delta_F)$ in a UP-algebra $\tilde{A}$ is called an intuitionistic fuzzy comparative UP-filter (IFCUPF) of $\tilde{A}$ if it satisfies* (11), (12), *and*

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) (\omega_F(\tilde{y}) \geq \min\{\omega_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \omega_F(\tilde{x})\}), \tag{17}$$

$$\left(\text{for all } \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}\right) (\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\}). \tag{18}$$

Example 3.1. *Consider a UP-algebra $\tilde{A} = \{0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4\}$ with the following Cayley table:*

$\star$	0	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$
0	0	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$
$\tilde{y}_1$	0	0	0	0	0
$\tilde{y}_2$	0	$\tilde{y}_2$	0	0	0
$\tilde{y}_3$	0	$\tilde{y}_2$	$\tilde{y}_4$	0	0
$\tilde{y}_4$	0	$\tilde{y}_1$	$\tilde{y}_3$	$\tilde{y}_3$	0

Define an IFS $F = (\omega_F, \delta_F)$ in $\tilde{A}$ as follows:

$\tilde{A}$	0	$\tilde{y}_1$	$\tilde{y}_2$	$\tilde{y}_3$	$\tilde{y}_4$
ω_F	0.2	0.7	0.2	0.8	0.2
δ_F	0.1	0.3	0.2	0.4	0.2

Then $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$.

Theorem 3.1. *Every IFCUPF of a UP-algebra $\tilde{A}$ is an IFUPF of $\tilde{A}$.*

Proof: Let $F = (\omega_F, \delta_F)$ be an IFCUPF of $\tilde{A}$. Then, for all $\tilde{x}, \tilde{y} \in \tilde{A}$, we have $\omega_F(0) \geq \delta_F(\tilde{x})$, $\delta_F(0) \leq \omega_F(\tilde{x})$, $\omega_F(\tilde{y}) \geq \min\{\omega_F(\tilde{x} \star ((\tilde{y} \star \tilde{y}) \star \tilde{y})), \omega_F(\tilde{x})\} = \min\{\omega_F(\tilde{x} \star \tilde{y}), \omega_F(\tilde{x})\}$, and $\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{y}) \star \tilde{y})), \delta_F(\tilde{x})\} = \max\{\delta_F(\tilde{x} \star \tilde{y}), \delta_F(\tilde{x})\}$. Hence, $F = (\omega_F, \delta_F)$ is an IFUPF of $\tilde{A}$. □

Example 3.2. *Consider a UP-algebra $\tilde{A} = \{0, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4\}$ with the following Cayley table:*

$\star$	0	$\tilde{z}_1$	$\tilde{z}_2$	$\tilde{z}_3$	$\tilde{z}_4$
0	0	$\tilde{z}_1$	$\tilde{z}_2$	$\tilde{z}_3$	$\tilde{z}_4$
$\tilde{z}_1$	0	0	$\tilde{z}_4$	$\tilde{z}_1$	$\tilde{z}_4$
$\tilde{z}_2$	0	$\tilde{z}_1$	0	$\tilde{z}_1$	0
$\tilde{z}_3$	0	0	$\tilde{z}_4$	0	$\tilde{z}_4$
$\tilde{z}_4$	0	$\tilde{z}_1$	$\tilde{z}_4$	$\tilde{z}_1$	0

Define an IFS $F = (\omega_F, \delta_F)$ in $\tilde{A}$ as follows:

$\tilde{A}$	0	$\tilde{z}_1$	$\tilde{z}_2$	$\tilde{z}_3$	$\tilde{z}_4$
ω_F	0.5	0.1	0.5	0.1	0.1
δ_F	0.4	0.3	0.4	0.3	0.3

Then $F = (\omega_F, \delta_F)$ is an IFUPF of $\tilde{A}$, but it is not an IFCUPF of $\tilde{A}$. Indeed, $\omega_F(\tilde{z}_3) = 0.1 < 0.5 = \min\{\omega_F(\tilde{z}_1 \star ((\tilde{z}_3 \star \tilde{z}_4) \star \tilde{z}_3)), \omega_F(\tilde{z}_1)\}$.

Theorem 3.2. *If an IFS $F = (\omega_F, \delta_F)$ in a UP-algebra $\tilde{A}$ is constant, then $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$.*

Proof: Suppose that an IFS $F = (\omega_F, \delta_F)$ in $\tilde{A}$ is constant. Then, there exist elements $\vec{m}$ and $\vec{n}$ in $[0, 1]$ such that $\omega_F(\tilde{x}) = \vec{m}$ and $\delta_F(\tilde{x}) = \vec{n}$ for all $\tilde{x} \in \tilde{A}$. Thus, $\omega_F(0) = \vec{m} = \omega_F(\tilde{x})$ and $\delta_F(0) = \vec{n} = \delta_F(\tilde{x})$ for all $\tilde{x} \in \tilde{A}$. For all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$, we get $\omega_F(\tilde{y}) = \vec{m} = \min\{\omega_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \omega_F(\tilde{x})\}$ and $\delta_F(\tilde{y}) = \vec{n} = \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\}$. Altogether, we have that $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. □

Theorem 3.3. *An IFS $F = (\omega_F, \delta_F)$ in a UP-algebra $\tilde{A}$ is an IFCUPF of $\tilde{A}$ if and only if the FSs ω_F and $\overline{\delta_F}$ are FCUPFs of $\tilde{A}$.*

Proof: Assume that an IFS $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. Clearly, ω_F is an FCUPF of $\tilde{A}$. Then, it is necessary to show that $\overline{\delta_F}$ is an FCUPF of $\tilde{A}$. Let $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$. By the assumption, we obtain $\delta_F(0) \leq \delta_F(\tilde{x})$ and $\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\}$. Thus, $\overline{\delta_F}(0) = 1 - \delta_F(0) \leq 1 - \delta_F(\tilde{x}) = \overline{\delta_F}(\tilde{x})$, so $\overline{\delta_F}(\tilde{y}) = 1 - \delta_F(\tilde{y}) \geq 1 - \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\} = \min\{1 - \delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), 1 - \delta_F(\tilde{x})\} = \min\left\{\overline{\delta_F}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \overline{\delta_F}(\tilde{x})\right\}$. Therefore, we have that $\overline{\delta_F}$ is an FCUPF of $\tilde{A}$.

Conversely, suppose that the FSs ω_F and $\overline{\delta_F}$ are FCUPFs of $\tilde{A}$. Clearly, $F = (\omega_F, \delta_F)$ satisfies (11) and (17). Since $\overline{\delta_F}$ is an FCUPF of $\tilde{A}$, we have $1 - \delta_F(0) \geq 1 - \delta_F(\tilde{x})$ and $1 - \delta_F(\tilde{y}) \geq \min\{1 - \delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), 1 - \delta_F(\tilde{x})\} = 1 - \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$. Thus, we illustrate that $\delta_F(0) \leq \delta_F(\tilde{x})$ and $\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$. This shows that $F = (\omega_F, \delta_F)$ satisfies (12) and (18). Therefore, $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. □

Theorem 3.4. *An IFS $F = (\omega_F, \delta_F)$ in a UP-algebra $\tilde{A}$ is an IFCUPF of $\tilde{A}$ if and only if the IFSs $\square F = (\omega_F, \overline{\omega_F})$ and $\diamond F = \left(\overline{\delta_F}, \delta_F\right)$ are IFCUPFs of $\tilde{A}$.*

Proof: Assume that $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. Then, we have $\square F = (\omega_F, \overline{\omega_F})$ satisfies (11) and (17). Thus, $\overline{\omega_F}(0) = 1 - \omega_F(0) \leq 1 - \omega_F(\tilde{x}) = \overline{\omega_F}(\tilde{x})$ and $\overline{\omega_F}(\tilde{y}) = 1 - \omega_F(\tilde{y}) \leq 1 - \min\{\omega_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \omega_F(\tilde{x})\} = \max\{1 - \omega_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), 1 - \omega_F(\tilde{x})\} = \max\{\overline{\omega_F}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \overline{\omega_F}(\tilde{x})\}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$. Hence, $\square F = (\omega_F, \overline{\omega_F})$ satisfies (12) and (18). Therefore, $\square F = (\omega_F, \overline{\omega_F})$ is an IFCUPF of $\tilde{A}$.

Next, we will show that $\diamond F = \left(\overline{\delta_F}, \delta_F\right)$ is an IFCUPF of $\tilde{A}$. By the assumption, we get $\diamond F = \left(\overline{\delta_F}, \delta_F\right)$ satisfies (12) and (18). Thus, for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$, $\overline{\delta_F}(0) = 1 - \delta_F(0) \leq 1 - \delta_F(\tilde{x}) = \overline{\delta_F}(\tilde{x})$ and $\overline{\delta_F}(\tilde{y}) = 1 - \delta_F(\tilde{y}) \geq 1 - \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\} = \min\{1 - \delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), 1 - \delta_F(\tilde{x})\} = \min\left\{\overline{\delta_F}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \overline{\delta_F}(\tilde{x})\right\}$. Hence, $\diamond F = \left(\overline{\delta_F}, \delta_F\right)$ satisfies (11) and (17). Therefore, $\diamond F = \left(\overline{\delta_F}, \delta_F\right)$ is an IFCUPF of $\tilde{A}$.

Assume that the IFSs $\square F = (\omega_F, \overline{\omega_F})$ and $\diamond F = \left(\overline{\delta_F}, \delta_F\right)$ are IFCUPFs of $\tilde{A}$. Then ω_F and $\overline{\delta_F}$ are FCUPFs of $\tilde{A}$. Therefore, it follows from Theorem 3.3 that $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. □

Theorem 3.5. *An IFS $F = (\omega_F, \delta_F)$ in a UP-algebra $\tilde{A}$ is an IFCUPF of $\tilde{A}$ if and only if the IFS $\triangle F = \left(\overline{\delta_F}, \overline{\omega_F}\right)$ is an IFCUPF of $\tilde{A}$.*

Proof: Assume that $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. By Theorem 3.4, we obtain that the IFSs $\square F = (\omega_F, \overline{\omega_F})$ and $\diamond F = \left(\overline{\delta_F}, \delta_F\right)$ are IFCUPFs of $\tilde{A}$. Thus, $\overline{\delta_F}$ satisfies (11) and (17), and $\overline{\omega_F}$ satisfies (12) and (18). Hence, $\triangle F = \left(\overline{\delta_F}, \overline{\omega_F}\right)$ is an IFCUPF of $\tilde{A}$.

Suppose that $\triangle F = \left(\overline{\delta_F}, \overline{\omega_F}\right)$ is an IFCUPF of $\tilde{A}$. By Theorem 3.3, we get that the FSs $\overline{\delta_F}$ and $\overline{\overline{\omega_F}}$ are FCUPFs of $\tilde{A}$. Since $\overline{\overline{\omega_F}} = 1 - (1 - \omega_F) = \omega_F$, we have ω_F is an FCUPF of A. By Theorem 3.3, we have $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. □

For a nonempty subset $\tilde{Z}$ of a nonempty set $\tilde{A}$, the characteristic function $f_{\tilde{Z}}$ of $\tilde{A}$ is a function of $\tilde{A}$ into $\{0, 1\}$ defined as follows: for all $\tilde{x} \in \tilde{A}$, $f_{\tilde{Z}}(\tilde{x}) = \begin{cases} 1 & \text{if } \tilde{x} \in \tilde{Z}, \\ 0 & \text{if } \tilde{x} \notin \tilde{Z}. \end{cases}$ Then, for all $\tilde{x} \in \tilde{A}$, we have $\overline{f_{\tilde{Z}}}(\tilde{x}) = \begin{cases} 1 & \text{if } \tilde{x} \notin \tilde{Z}, \\ 0 & \text{if } \tilde{x} \in \tilde{Z}. \end{cases}$ Now, we denote the IFS in $\tilde{A}$ with the degree of membership $f_{\tilde{Z}}$ and the degree of nonmembership $\overline{f_{\tilde{Z}}}$ by $F_{\tilde{Z}}$, that is, $F_{\tilde{Z}} = \left(f_{\tilde{Z}}, \overline{f_{\tilde{Z}}}\right)$.

Theorem 3.6. *A nonempty subset $\tilde{Z}$ of a UP-algebra $\tilde{A}$ is a CUPF of $\tilde{A}$ if and only if the IFS $F_{\tilde{Z}} = \left(f_{\tilde{Z}}, \overline{f_{\tilde{Z}}}\right)$ is an IFCUPF of $\tilde{A}$.*

Proof: Assume that $\tilde{Z}$ is a CUPF of $\tilde{A}$. Then, since $0 \in \tilde{Z}$, we have $f_{\tilde{Z}}(0) = 1 \geq f_{\tilde{Z}}(\tilde{x})$ and $\overline{f_{\tilde{Z}}}(0) = 0 \leq \overline{f_{\tilde{Z}}}(\tilde{x})$ for all $\tilde{x} \in \tilde{A}$. Thus, $F_{\tilde{Z}}$ satisfies (11) and (12). Let $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$. In the case that $\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y}) \notin \tilde{Z}$ or $\tilde{x} \notin \tilde{Z}$, we have $\overline{f_{\tilde{Z}}}(\tilde{y}) \leq 1 = \max\left\{\overline{f_{\tilde{Z}}}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \overline{f_{\tilde{Z}}}(\tilde{x})\right\}$ and $f_{\tilde{Z}}(\tilde{y}) \geq 0 = \min\{f_{\tilde{Z}}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), f_{\tilde{Z}}(\tilde{x})\}$. On the other hand, let $\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y}) \in \tilde{Z}$ and $\tilde{x} \in \tilde{Z}$. By the assumption, we get $\tilde{y} \in \tilde{Z}$. Hence, $\overline{f_{\tilde{Z}}}(\tilde{y}) = 0 = \max\left\{\overline{f_{\tilde{Z}}}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \overline{f_{\tilde{Z}}}(\tilde{x})\right\}$ and $f_{\tilde{Z}}(\tilde{y}) = 1 = \min\{f_{\tilde{Z}}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), f_{\tilde{Z}}(\tilde{x})\}$. This shows that $F_{\tilde{Z}}$ satisfies (17) and (18). Therefore, $F_{\tilde{Z}}$ is an IFCUPF of $\tilde{A}$.

Assume that $F_{\tilde{Z}} = \left(f_{\tilde{Z}}, \overline{f_{\tilde{Z}}}\right)$ is an IFCUPF of $\tilde{A}$. Then $f_{\tilde{Z}}(0) \geq f_{\tilde{Z}}(\tilde{x}) = 1$ when $\tilde{x} \in \tilde{Z}$. Thus, $0 \in \tilde{Z}$ and so $\tilde{Z}$ satisfies (2). Next, we will show that $\tilde{Z}$ satisfies (5). Let $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{A}$ be such that $\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y}) \in \tilde{Z}$ and $\tilde{x} \in \tilde{Z}$. By using the assumption, we get $f_{\tilde{Z}}(\tilde{y}) \geq \min\{f_{\tilde{Z}}(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), f_{\tilde{Z}}(\tilde{x})\} = 1$. Hence, $\tilde{y} \in \tilde{Z}$. This shows that $\tilde{Z}$ satisfies (5). Altogether, $\tilde{Z}$ is a CUPF of $\tilde{A}$. □

Definition 3.2. *Let ω and δ be FSs in a nonempty set $\tilde{A}$. For $\vec{m},\vec{n} \in [0,1]$, the set* $\tilde{U}(\omega;\vec{m}) = \left\{\tilde{x} \in \tilde{A} \middle| \omega(\tilde{x}) \geq \vec{m}\right\}$ *and* $\tilde{U}^{+}(\omega;\vec{m}) = \left\{\tilde{x} \in \tilde{A} \middle| \omega(\tilde{x}) > \vec{m}\right\}$ *are called an* upper $\vec{m}$-level subset *and an* upper $\vec{m}$-strong level subset *of ω, respectively. The set* $\tilde{L}(\omega;\vec{m}) = \left\{\tilde{x} \in \tilde{A} \middle| \omega(\tilde{x}) \leq \vec{m}\right\}$ *and* $\tilde{L}^{-}(\omega;\vec{m}) = \left\{\tilde{x} \in \tilde{A} \middle| \omega(\tilde{x}) < \vec{m}\right\}$ *are called a* lower $\vec{m}$-level subset *and a* lower $\vec{m}$-strong level subset *of ω, respectively. The set* $\tilde{C}(\omega,\delta;\vec{m},\vec{n}) = \tilde{U}(\omega;\vec{m}) \cap \tilde{L}(\delta,\vec{n})$ *is called the* $(\vec{m},\vec{n})$-cut *of ω and δ.*

Theorem 3.7. *An IFS $F = (\omega_F, \delta_F)$ in a UP-algebra $\tilde{A}$ is an IFCUPF of $\tilde{A}$ if and only if the sets $\tilde{U}(\omega_F;\vec{m})$ and $\tilde{L}(\delta_F;\vec{n})$ are CUPFs of $\tilde{A}$ for each $\vec{m},\vec{n} \in [0,1]$ such that $\tilde{U}(\omega_F;\vec{m}) \neq \emptyset$ and $\tilde{L}(\delta_F;\vec{n}) \neq \emptyset$.*

Proof: Assume that $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$, let $\vec{m},\vec{n} \in [0,1]$ be such that $\tilde{U}(\omega_F;\vec{m})$ and $\tilde{L}(\delta_F;\vec{n})$ are nonempty subsets of $\tilde{A}$. Then, there exist $\tilde{x} \in \tilde{U}(\omega_F;\vec{m})$ and $\tilde{y} \in \tilde{L}(\delta_F;\vec{n})$. Thus, $\omega(\tilde{x}) \geq \vec{m}$ and $\delta_F(\tilde{y}) \leq \vec{n}$. By the assumption, we have $\omega_F(0) \geq \omega_F(\tilde{x}) \geq \vec{m}$ and $\delta_F(0) \leq \delta_F(\tilde{y}) \leq \vec{n}$. Hence, $0 \in \tilde{U}(\omega_F;\vec{m})$ and $0 \in \tilde{L}(\delta_F;\vec{n})$. Therefore, $\tilde{U}(\omega_F;\vec{m})$ and $\tilde{L}(\delta_F;\vec{n})$ satisfy (2).

Next, will show that $\tilde{U}(\omega_F;\vec{m})$ satisfies (5). Let $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$ be such that $\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y}) \in \tilde{U}(\omega;\vec{m})$ and $\tilde{x} \in \tilde{U}(\omega;\vec{m})$. Thus, $\omega_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})) \geq \vec{m}$ and $\omega_F(\tilde{x}) \geq \vec{m}$. By the assumption, we have $\omega_F(\tilde{y}) \geq \min\{\omega_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \omega_F(\tilde{x})\} \geq \vec{m}$. So, $y \in \tilde{U}(\omega_F;\vec{m})$. Hence, $\tilde{U}(\omega_F;\vec{m})$ satisfies (5).

Finally, to show that $\tilde{L}(\delta_F;\vec{n})$ satisfies (5), let $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$ be such that $\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y}) \in \tilde{L}(\delta_F;\vec{n})$ and $\tilde{x} \in \tilde{L}(\delta_F;\vec{n})$. Then $\delta_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})) \leq \vec{n}$ and $\delta_F(\tilde{x}) \leq \vec{n}$. Thus, $\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \delta_F(\tilde{x})\} \leq \vec{n}$. This shows that $y \in \tilde{L}(\delta_F;\vec{n})$. Hence, $\tilde{L}(\delta_F;\vec{n})$ satisfies (5).

Altogether, we have that $\tilde{U}(\omega_F;\vec{m})$ and $\tilde{L}(\delta_F,\vec{n})$ are CUPFs of $\tilde{A}$.

Assume that the sets $\tilde{U}(\omega_F;\vec{m})$ and $\tilde{L}(\delta_F;\vec{n})$ are CUPFs of $\tilde{A}$ for each $\vec{m},\vec{n} \in [0,1]$ such that $\tilde{U}(\omega_F;\vec{m}) \neq \emptyset$ and $\tilde{L}(\delta_F;\vec{n}) \neq \emptyset$. Let $\tilde{x} \in \tilde{A}$. Then, we have $\tilde{x} \in \tilde{U}(\omega_F;\omega_F(\tilde{x}))$ and $\tilde{x} \in \tilde{L}(\delta_F;\delta_F(\tilde{x}))$. By the assumption, we have $\tilde{U}(\omega_F;\omega_F(\tilde{x}))$ and $\tilde{L}(\delta_F;\delta_F(\tilde{x}))$ are CUPFs of $\tilde{A}$. Thus, $0 \in \tilde{U}(\omega_F;\omega_F(\tilde{x}))$ and $0 \in \tilde{L}(\delta_F;\delta_F(\tilde{x}))$ which imply $\omega_F(0) \geq \omega_F(\tilde{x})$ and $\delta_F(0) \leq \delta_F(\tilde{x})$. Hence, $F = (\omega_F, \delta_F)$ satisfies (11) and (12).

Next, we will show that $F = (\omega_F, \delta_F)$ satisfies (17). Suppose that there exist $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$ such that $\omega_F(\tilde{y}) < \min\{\omega_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \omega_F(\tilde{x})\}$. Choose $\vec{m}_0 = \frac{1}{2}[\omega_F(\tilde{y}) + \min\{\omega_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \omega_F(\tilde{x})\}]$. Thus, $\vec{m}_0 \in [0,1]$ and $\omega_F(\tilde{y}) < \vec{m}_0 < \min\{\omega_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \omega_F(\tilde{x})\}$. It implies that $\tilde{y} \notin \tilde{U}(\omega_F;\vec{m}_0)$ but $\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y}), \tilde{x} \in \tilde{U}(\omega_F;\vec{m}_0)$. Thus, $\tilde{U}(\omega_F;\vec{m}_0)$ is not a CUPF of $\tilde{A}$, which is a contradiction. Hence, we obtain that $\omega_F(\tilde{y}) \geq \min\{\omega_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \omega_F(\tilde{x})\}$ for all $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$. This implies that $F = (\omega_F, \delta_F)$ satisfies (17).

Finally, we will show that $F = (\omega_F, \delta_F)$ satisfies (18). Suppose that there exist $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$ such that $\delta_F(\tilde{y}) > \max\{\delta_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \delta_F(\tilde{x})\}$. Choose $\vec{n}_0 = \frac{1}{2}[\delta_F(\tilde{y}) + \max\{\delta_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \delta_F(\tilde{x})\}]$. Thus, $\vec{n}_0 \in [0,1]$ and $\max\{\delta_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \delta_F(\tilde{x})\} < \vec{n}_0 < \delta_F(\tilde{y})$. It implies that $\tilde{y} \in \tilde{L}(\delta_F;\vec{n}_0)$ but $\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})$, $\tilde{x} \in \tilde{L}(\delta_F;\vec{n}_0)$. Then, $\tilde{L}(\delta_F;\vec{n}_0)$ is not a CUPF of $\tilde{A}$, which is a contradiction. Therefore, $\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x}\star((\tilde{y}\star\tilde{z})\star\tilde{y})), \delta_F(\tilde{x})\}$ for all $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$, which implies that $F = (\omega_F, \delta_F)$ satisfies (18).

Altogether, we get that $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$. □

Corollary 3.1. *An IFS $F = (\omega_F, \delta_F)$ is an IFCUPF of $\tilde{A}$ if and only if, for all $\vec{m},\vec{n} \in [0,1]$, the set $\tilde{C}(\omega_F,\delta_F;\vec{m},\vec{n})$ is either empty or a CUPF of $\tilde{A}$.*

Proof: The necessity is straightforward from Theorems 2.1 and 3.7.

Conversely, assume that, the set $\tilde{C}(\omega_F,\delta_F;\vec{m},\vec{n})$ is either empty or a CUPF of $\tilde{A}$ for all $\vec{m},\vec{n} \in [0,1]$. Let $\vec{m} \in [0,1]$ be such that $\tilde{U}(\omega_F;\vec{m}) \neq \emptyset$. Then $\emptyset \neq \tilde{U}(\omega_F;\vec{m}) =$

$\tilde{U}(\omega_F;\vec{m}) \cap \tilde{A} = \tilde{U}(\omega_F;\vec{m}) \cap \tilde{L}(\delta_F;1) = \tilde{C}(\omega_F,\delta_F;\vec{m},1)$. By the assumption, we have that $\tilde{U}(\omega_F;\vec{m}) = \tilde{C}(\omega_F,\delta_F;\vec{m},1)$ is a CUPF of $\tilde{A}$.

Let $\vec{n} \in [0,1]$ be such that $\tilde{L}(\delta_F;s) \neq \emptyset$. Then $\emptyset \neq \tilde{L}(\delta_F;\vec{n}) = \tilde{A} \cap \tilde{L}(\delta_F;\vec{n}) = \tilde{U}(\omega_F;0) \cap \tilde{L}(\delta_F;\vec{n}) = \tilde{C}(\omega_F,\delta_F;0,\vec{n})$. By the assumption, we get that $\tilde{L}(\delta_F;\vec{n}) = \tilde{C}(\omega_F,\delta_F;0,\vec{n})$ is a CUPF of $\tilde{A}$. By Theorem 3.7, we have $F = (\omega_F,\delta_F)$ is an IFCUPF of $\tilde{A}$. □

Theorem 3.8. *If an IFS $F = (\omega_F,\delta_F)$ in a UP-algebra $\tilde{A}$ is an IFCUPF of $\tilde{A}$, then the set $\tilde{U}^+(\omega_F;\vec{m})$ and $\tilde{L}^-(\delta_F;\vec{n})$ are CUPFs of $\tilde{A}$ for each $\vec{m},\vec{n} \in [0,1]$ such that $\tilde{U}^+(\omega_F;\vec{m}) \neq \emptyset$ and $\tilde{L}^-(\delta_F;\vec{n}) \neq \emptyset$.*

Proof: Suppose that an IFS $F = (\omega_F,\delta_F)$ is an IFCUPF of $\tilde{A}$. Let $\vec{m},\vec{n} \in [0,1]$ be such that $\tilde{U}^+(\omega_F;\vec{m})$ and $\tilde{L}^-(\delta_F;\vec{n})$ are nonempty subsets of $\tilde{A}$. Then, there exist $\tilde{a} \in \tilde{U}^+(\omega_F;\vec{m})$ and $\tilde{b} \in \tilde{L}^-(\delta_F;\vec{n})$ which imply that $\omega_F(\tilde{y}) > \vec{m}$ and $\delta_F(\tilde{z}) < \vec{n}$. By the assumption, we have $\omega_F(0) \geq \omega_F(\tilde{y}) > \vec{m}$ and $\delta_F(0) \leq \delta_F(\tilde{z}) < \vec{n}$. Thus, $0 \in \tilde{U}^+(\omega_F;\vec{m})$ and $0 \in \tilde{L}(\delta_F;\vec{n})$. Hence, $\tilde{U}^+(\omega_F;\vec{m})$ and $\tilde{L}^-(\delta_F;\vec{n})$ satisfy (2).

Next, let $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$ be such that $\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star y)$, $\tilde{x} \in \tilde{U}^+(\omega_F;\vec{m})$. Then, $\omega_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star y)) > \vec{m}$ and $\omega_F(\tilde{x}) > \vec{m}$. Thus, $\omega_F(\tilde{y}) \geq \min\{\omega_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star y)), \omega_F(\tilde{x})\} > \vec{m}$, which implies $\tilde{y} \in \tilde{U}^+(\omega_F;\vec{m})$. Hence, $\tilde{U}^+(\omega_F;\vec{m})$ satisfies (5).

Finally, let $\tilde{x},\tilde{y},\tilde{z} \in \tilde{A}$ be such that $\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y}) \in \tilde{L}^-(\delta_F;\vec{n})$ and $\tilde{x} \in \tilde{L}^-(\delta_F;\vec{n})$. Then $\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})) < \vec{n}$ and $\delta_F(\tilde{x}) < \vec{n}$. Thus, $\delta_F(\tilde{y}) \leq \max\{\delta_F(\tilde{x} \star ((\tilde{y} \star \tilde{z}) \star \tilde{y})), \delta_F(\tilde{x})\} < \vec{n}$, so $\tilde{y} \in \tilde{L}^-(\delta_F;\vec{n})$. This shows that $\tilde{L}^-(\delta_F;\vec{n})$ satisfies (5).

Altogether, $\tilde{U}^+(\omega_F;\vec{m})$ and $\tilde{L}^-(\delta_F;\vec{n})$ are CUPFs of $\tilde{A}$. □

4. Conclusion.

In this paper, we have introduced the concept of IFCUPF of UP-algebras, and provided their properties. In our future work, we will utilize ideas and results in this paper in IFCUPFs to study the substructures of algebraic systems related to UP-algebras.

In the near future, we will broaden the scope of the research covered in this work to include investigation into essential comparative UP-filters and t-essential intuitionistic fuzzy comparative UP-filters, in accordance with [22, 23].

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