ICIC Express Letters
Volume **17**, Number **2**, February **2023**

ICIC International ©2023 ISSN 1881-803X
pp. 191–199

ESSENTIAL IDEALS AND THEIR FUZZIFICATIONS OF TERNARY SEMIGROUPS

Rattiya Rittichuai[1], Aiyared Iampan[2], Ronnason Chinram[3] and Pattarawan Singavananda[4,*]

[1]Department of Mathematics
Faculty of Science and Technology
Nakhon Si Thammarat Rajabhat University
Tha Ngio, Mueang, Nakhon Si Thammarat 80280, Thailand
rattiya_ari@nstru.ac.th

[2]Department of Mathematics
School of Science
University of Phayao
Maeka, Muang, Phayao 56000, Thailand
aiyared.ia@up.ac.th

[3]Division of Computational Science
Faculty of Science
Prince of Songkla University
Karnjanavanich Road, Hat Yai, Songkhla 90110, Thailand
ronnason.c@psu.ac.th

[4]Program in Mathematics
Faculty of Science and Technology
Songkhla Rajabhat University
Khoa-Roob-Chang, Muang, Songkhla 90000, Thailand
*Corresponding author: pattarawan.pe@skru.ac.th

Received March 2022; accepted June 2022

Abstract. *In this paper, we introduce the notions of essential ideals and* 0*-essential ideals of ternary semigroups and the concept of their fuzzifications. This study examines relationships between essential ideals and their fuzzifications and relationships between* 0*-essential ideals and their fuzzications. In addition, we propose the concept of minimal (prime, semiprime) essential ideals and minimal (prime, semiprime) essential fuzzy ideals of ternary semigroups and provide the relationships between them. Moreover, we investigate in the same manner in the case of* 0*-essential ideals and* 0*-essential fuzzy ideals.*

Keywords: Essential ideals, 0-essential ideals, Essential fuzzy ideals, 0-essential fuzzy ideals, Maximal, Prime, Semiprime

1. **Introduction.** A ternary semigroup is a nonempty set equipped with an associative ternary operation so that any semigroup can be reduced to a ternary semigroup but a ternary semigroup does not necessarily reduce to a semigroup [1, 2]. One knows that ideal theories play an important role in many algebraic structures. The classical of fuzzy sets was introduced in 1965 by Zadeh [3] and after that the concept of fuzzy sets has been educated in various fields. The notion of fuzzy semigroups was extended to many algebraic structures, and one of those is the concept of fuzzy ternary semigroups [4, 5, 6, 7]. The concept of essential fuzzy ideals of rings was explored by Medhi et al. in 2008 [8]. Later in 2013, Medhi and Saikia [9] investigated the notion of T-fuzzy essential ideals of rings. In 2021, Baupradist et al. [10] examined essential ideals and essential fuzzy ideals in

DOI: 10.24507/icicel.17.02.191

semigroups. Moreover, they also studied 0-essential ideals and 0-essential fuzzy ideals in semigroups. Later, essential (m, n)-ideals and essential fuzzy (m, n)-ideals in semigroups were proposed in [11]. Recently, essential UP-ideals and t-essential fuzzy UP-ideals of UP-algebras were found out in [12].

The purposes of this study were to investigate essential ideals and essential fuzzy ideals in ternary semigroups. Our main results are divided into the following sections. In Section 2, we provide definitions that we will be used throughout this paper. In Section 3, we focus our work on essential ideals and essential fuzzy ideals in ternary semigroups. In Section 4, we investigate 0-essential ideals and 0-essential fuzzy ideals of ternary semigroups with zero in the same manner as Section 3. In the final section, Section 5, we summarize the main results of our work and give the suggestion for further research.

2. Preliminaries.

In this section, we recall some basic concepts that will be used throughout this paper.

2.1. Ternary semigroups.

Definition 2.1. *A **ternary semigroup** is a nonempty set T together with a ternary operation $(a, b, c) \mapsto [abc]$ satisfying the associative law*

$$[[abc]uv] = [a[bcu]v] = [ab[cuv]] \text{ for all } a, b, c, u, v \in T.$$

Example 2.1. *Let $\mathbb{Z}^-$ be the set of all negative integers. We have that $\mathbb{Z}^-$ is a ternary semigroup under multiplication over integer numbers. It is easy to see that $\mathbb{Z}^-$ is not a binary semigroup under multiplication over integer numbers.*

For three nonempty subsets A, B and C of a ternary semigroup T, let

$$[ABC] = \{[abc] \mid a \in A, b \in B, c \in C\}.$$

Definition 2.2. *Let T be a ternary semigroup. An element 0 in T is called a **zero** of T if $[0xy] = [x0y] = [xy0] = 0$ for all $x, y \in T$, and we call T a ternary semigroup with zero.*

Definition 2.3. *Let T be a ternary semigroup. A non-empty subset A of T is called*

(1) *a **ternary subsemigroup** of T if $[AAA] \subseteq A$,*
(2) *a **left ideal** of T if $[TTA] \subseteq A$,*
(3) *a **right ideal** of T if $[ATT] \subseteq A$,*
(4) *a **lateral ideal** of T if $[TAT] \subseteq A$,*
(5) *an **ideal** if A is a left, right and lateral ideal of T.*

Definition 2.4. *Let X be a nonempty subset of a ternary semigroup T. The **ideal of** T **generated by** X is the intersection of all ideals of T containing X, denoted by $(X)_i$.*

For any non-empty set X of a ternary semigroup T, we have that

$$(X)_i = X \cup [XTT] \cup [TTX] \cup [TXT] \cup [TTXTT].$$

2.2. Fuzzy subsets.

A fuzzy subset of a set S is a membership function from S into $[0, 1]$. A set S is also a fuzzy subset by $S(x) = 1$ for all $x \in S$. For any two fuzzy subsets f and g of S,

1) $f \cap g$ is a fuzzy subset of S defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\}$$

for all $x \in S$,

2) $f \cup g$ is a fuzzy subset of S defined by

$$(f \cup g)(x) = \max\{f(x), g(x)\}$$

for all $x \in S$ and

3) $f \subseteq g$ if $f(x) \leq g(x)$ for all $x \in S$.

For a fuzzy subset f of a set S, the support of f is defined by

$$supp(f) = \{x \in S \mid f(x) \neq 0\}.$$

The characteristic mapping of a subset A of a set S is a fuzzy subset of S defined by

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

The concept of fuzzy semigroups was extended to many algebraic structures, and one of those is the concept of fuzzy ternary semigroups which were defined as follows.

Definition 2.5. *A fuzzy subset f of a ternary semigroup T is called a **fuzzy ideal** of T if $f([xyz]) \geq \max\{f(x), f(y), f(z)\}$ for all $x, y, z \in T$.*

Definition 2.6. *For any three fuzzy subsets f_1, f_2 and f_3 of a ternary semigroup T. The **product** $[f_1 \circ f_2 \circ f_3]$ of f_1, f_2 and f_3 is defined by*

$$[f_1 \circ f_2 \circ f_3](y) = \begin{cases} \sup_{y=[y_1y_2y_3]} \min\{f_1(y_1), f_2(y_2), f_3(y_3)\} & \text{if } y \in [TTT], \\ 0 & \text{otherwise.} \end{cases}$$

Then a fuzzy subset f of a ternary semigroup T is a fuzzy ternary subsemigroup of T if and only if $[f \circ f \circ T] \subseteq f$, $[f \circ T \circ f] \subseteq f$ and $[T \circ f \circ f] \subseteq f$.

3. Essential Ideals and Essential Fuzzy Ideals.

In this section, the algebraic structures namely essential ideals and essential fuzzy ideals of ternary semigroups will be presented.

Definition 3.1. *An ideal I of a ternary semigroup T is called an **essential ideal** of T if $I \cap J \neq \emptyset$ for every ideal J of T.*

Example 3.1. (1) *We have that T is an essential ideal of a ternary semigroup T.*

(2) *Let T be a ternary semigroup with zero. So for every ideal I and J, we have $0 \in I \cap J$. Therefore, every ideal of T is an essential ideal of T.*

Proposition 3.1. *Let I be an essential ideal of a ternary semigroup T. If I' is any ideal of T containing I, then I' is also an essential ideal of T.*

Proof: Assume that I is an essential ideal of T and let I' be any ideal of T such that $I \subseteq I'$. So $I \cap J \subseteq I' \cap J$ and $I \cap J \neq \emptyset$ for every ideal J of T. This implies that $I' \cap J \neq \emptyset$ for every ideal J of T. Therefore, I' is an essential ideal of T. □

Proposition 3.2. *Let T be a ternary semigroup. The following statements are true.*

(1) *The union of any two essential ideals of T is also an essential ideal of T.*
(2) *The intersection of any two essential ideals of T is also an essential ideal of T.*

Proof: Assume that I_1 and I_2 are any two essential ideals of a ternary semigroup T.

(1) Since $I_1 \subseteq I_1 \cup I_2$ and I_1 is an essential ideal of T, by Proposition 3.1, we have $I_1 \cup I_2$ is an essential ideal of T.

(2) Let J be any ideal of T. Thus, $I_1 \cap J \neq \emptyset$. So there exists $x \in I_1 \cap J$. Let $y \in I_2$. Then $[xxy] \in (I_1 \cap I_2) \cap J$. Therefore, $(I_1 \cap I_2) \cap J \neq \emptyset$. Hence, $I_1 \cap I_2$ is an essential ideal of T. □

Next, we extend the notion of essential ideals of ternary semigroups to the concept of essential fuzzy ideals of ternary semigroups that will be defined as follows.

Definition 3.2. *A nonzero fuzzy ideal f of a ternary semigroup T is called an **essential fuzzy ideal** of T if $f \cap g \neq 0$ for every nonzero fuzzy ideal g of T.*

The two following theorems investigate relationships between essential ideals and essential fuzzy ideals of a ternary semigroup.

Theorem 3.1. *An ideal I of a ternary semigroup T is essential if and only if χ_I is an essential fuzzy ideal of T.*

Proof: Assume that I is an essential ideal of a ternary semigroup T and let g be any nonzero fuzzy ideal of T. Then $supp(g)$ is an ideal of T. Thus, $I \cap supp(g) \neq \emptyset$. So there exists $x \in S$ such that $x \in I \cap supp(g)$, and this implies that $(\chi_I \cap g)(x) \neq 0$. Therefore, $\chi_I \cap g \neq 0$. Hence, χ_I is an essential fuzzy ideal of T. Conversely, assume that χ_I is an essential fuzzy ideal of T. Let J be any ideal of T. Thus, χ_J is a nonzero fuzzy ideal of T. Then $\chi_I \cap \chi_J \neq 0$. It implies that $\chi_{I\cap J} \neq 0$. Hence, $I \cap J \neq \emptyset$. □

Theorem 3.2. *A nonzero fuzzy ideal f of a ternary semigroup T is essential if and only if $supp(f)$ is an essential ideal of T.*

Proof: Assume that f is essential. Since f is a fuzzy ideal of T, $supp(f)$ is an ideal of T. Let J be any ideal of T. Then $f \cap \chi_J \neq 0$. So there exists $x \in T$ such that $(f \cap \chi_J)(x) \neq 0$. Thus, $f(x) \neq 0$ and $\chi_J(x) \neq 0$. Hence, $x \in supp(f) \cap J$. This implies that $supp(f) \cap J \neq \emptyset$, so $supp(f)$ is an essential ideal of T. Conversely, assume that $supp(f)$ is an essential ideal of T. Let g be a nonzero fuzzy ideal of T. Then $supp(g)$ is an ideal of T and so $supp(f) \cap supp(g) \neq \emptyset$. Thus, there exists $x \in S$ such that $x \in supp(f) \cap supp(g)$, so $f(x) \neq 0$ and $g(x) \neq 0$. Therefore, $(f \cap g)(x) \neq 0$. Hence, $f \cap g \neq 0$. Therefore, f is essential. □

Proposition 3.3. *Let f be an essential fuzzy ideal of a ternary semigroup T. If f' is a fuzzy ideal of T such that $f \subseteq f'$, then f' is also essential.*

Proof: Assume that f is an essential fuzzy ideal of T and let f' be a fuzzy ideal of T such that $f \subseteq f'$. Let g be any fuzzy ideal of T. Thus, $f \cap g \neq 0$. So $f' \cap g \neq 0$. Hence, f' is essential. □

Theorem 3.3. *Let T be a ternary semigroup. The following statements are true.*

(1) *The union of any two essential fuzzy ideals of T is an essential fuzzy ideal of T.*
(2) *The intersection of any two essential fuzzy ideals of T is an essential fuzzy ideal of T.*

Proof: (1) It follows by Proposition 3.3.

(2) Assume that f_1 and f_2 are essential fuzzy ideals of T. Then $f_1 \cap f_2$ is a fuzzy ideal of T. Let g be a nonzero fuzzy ideal of T. Then $f_1 \cap g \neq 0$. Thus, there exists $x \in S$ such that $(f_1 \cap g)(x) \neq 0$. Then $f_1(x) \neq 0$ and $g(x) \neq 0$. Since $f_2 \neq 0$, let $y \in S$ be such that $f_2(y) \neq 0$. Since f_1 and f_2 are fuzzy ideals of T,

$$f_1([xxy]) \geq \max\{f_1(x), f_1(y)\} \geq f_1(x) > 0$$

and

$$f_2([xxy]) \geq \max\{f_2(x), f_2(y)\} \geq f_2(y) > 0.$$

So $(f_1 \cap f_2)([xxy]) = \min\{f_1([xxy]), f_2([xxy])\} \neq 0$. Since g is a fuzzy ideal of T and $g(x) \neq 0$, $g([xxy]) \neq 0$. So $[(f_1 \cap f_2) \cap g]([xxy]) \neq 0$. Therefore, $[(f_1 \cap f_2) \cap g] \neq 0$. This implies that $f_1 \cap f_2$ is an essential fuzzy ideal of T. □

Next, we propose the concept of minimal (prime, semiprime) essential ideals and minimal (prime, semiprime) essential fuzzy ideals of ternary semigroups.

Definition 3.3. *An essential ideal I of a ternary semigroup T is called **minimal** if for every essential ideal J of T such that $J \subseteq I$, we have $J = I$.*

Example 3.2. *Let T be a ternary semigroup with zero. Then $\{0\}$ is a unique minimal essential ideal of T.*

Definition 3.4. *An essential fuzzy ideal f of a ternary semigroup T is called **minimal** if for every essential fuzzy ideal g of T such that $g \subseteq f$, we have $supp(f) = supp(g)$.*

In the remainder of this section, we provide the relationships between minimal (prime, semiprime) essential ideals and minimal (prime, semiprime) essential fuzzy ideals of ternary semigroups.

Theorem 3.4. *A nonempty subset A of a ternary semigroup T is a minimal essential ideal of T if and only if χ_A is a minimal essential fuzzy ideal of T.*

Proof: Assume that A is a minimal essential ideal of T. By Theorem 3.1, we have χ_A is an essential fuzzy ideal of T. Let f be any essential fuzzy ideal of T such that $f \subseteq \chi_A$. Thus, $supp(f) \subseteq supp(\chi_A) = A$. By Theorem 3.2, $supp(f)$ is an essential ideal of T. So $supp(f) = A$ because A is minimal. Therefore, $supp(f) = supp(\chi_A)$. Hence, χ_A is minimal.

To prove the converse, assume that χ_A is a minimal essential fuzzy ideal of T and let B be an essential fuzzy ideal of T such that $B \subseteq A$. So χ_B is an essential fuzzy ideal of T such that $\chi_B \subseteq \chi_A$. Therefore, $B = supp(\chi_B) = supp(\chi_A) = A$. Hence, A is minimal. □

Definition 3.5. *Let T be any ternary semigroup.*

(1) *An essential ideal A of T is called **prime** if*

$$[xyz] \in A \textit{ implies } x \in A \textit{ or } y \in A \textit{ or } z \in A \textit{ for all } x, y, z \in T.$$

(2) *An essential fuzzy ideal f of T is called **prime** if*

$$f([xyz]) \leq \max\{f(x), f(y), f(z)\} \textit{ for all } x, y, z \in T.$$

Theorem 3.5. *A nonempty subset A of a ternary semigroup T is a prime essential ideal of T if and only if χ_A is a prime essential fuzzy ideal of T.*

Proof: Assume that A is a prime essential ideal of T. We have χ_A is an essential fuzzy ideal of T by Theorem 3.1. Let x, y and z be any three elements of T. If $[xyz] \in A$, then we have that $x \in A$ or $y \in A$ or $z \in A$ because A is prime. Thus, $\max\{\chi_A(x), \chi_A(y), \chi_A(z)\} = 1 \geq \chi_A([xyz])$. Otherwise, if $[xyz] \notin A$, then we have $\chi_A([xyz]) = 0 \leq \max\{\chi_A(x), \chi_A(y), \chi_A(z)\}$. By both two cases, we conclude that χ_A is a prime essential fuzzy ideal of T. Conversely, assume that χ_A is a prime essential ideal of T. It follows by Theorem 3.1 that A is an essential ideal of T. Let x, y and z be any three elements of T such that $[xyz] \in A$. This implies that $\chi_A([xyz]) = 1$. Since A is prime, $\chi_A([xyz]) \leq \max\{\chi_A(x), \chi_A(y), \chi_A(z)\}$. So $\max\{\chi_A(x), \chi_A(y), \chi_A(z)\}$ must be equal to 1 and thus $x \in A$ or $y \in A$ or $z \in A$. Hence, A is a prime essential ideal of T. □

Definition 3.6. *Let T be a ternary semigroup.*

(1) *An essential ideal A of T is called **semiprime** if for all $x \in T$, $[xxx] \in A$ implies $x \in A$.*
(2) *An essential fuzzy ideal f is called **semiprime** if for all $x \in T$, $f([xxx]) \leq f(x)$.*

Theorem 3.6. *A nonempty subset A of a ternary semigroup T is a semiprime essential ideal of T if and only if χ_A is a semiprime fuzzy essential fuzzy ideal of T.*

Proof: Assume that A is a semiprime essential ideal of T. By Theorem 3.1, we have χ_A is an essential fuzzy ideal of T. Let x be any element of T. If $[xxx] \in A$, then we have $x \in A$ because A is semiprime. This implies that $\chi_A(x) = 1$. Hence, $\chi_A(x) \geq \chi_A([xxx])$. Otherwise, if $[xxx] \notin A$, then we have $\chi_A([xxx]) = 0 \leq \chi_A(x)$. By both two cases, we conclude that χ_A is a semiprime essential fuzzy ideal of T. Conversely, assume that χ_A is a semiprime essential fuzzy ideal of T. By Theorem 3.1, we have that A is an essential ideal of T. Let $x \in T$ such that $[xxx] \in A$. So $\chi_A([xxx]) = 1$. Since χ_A is semiprime, $\chi_A([xxx]) \leq \chi_A(x)$. Since $\chi_A([xxx]) = 1$, $\chi_A(x)$ must be equal to 1. Therefore, $x \in A$. Hence, A is a semiprime essential ideal of T. □

4. **0-Essential Ideals and 0-Essential Fuzzy Ideals.** In this section, we focus our results on 0-essential ideals and 0-essential fuzzy ideals of a ternary semigroup with zero.

Definition 4.1. *Let T be a ternary semigroup with zero. A nonzero ideal I of T is called a* ***0-essential ideal*** *of T if $I \cap J \neq \{0\}$ for every nonzero ideal J of T.*

Example 4.1. *Consider the ternary semigroup $(\mathbb{Z}_{12}, +)$. We have that $\{0, 2, 4, 6, 8, 10\}$ and $\mathbb{Z}_{12}$ are only 0-essential ideals of $\mathbb{Z}_{12}$.*

Proposition 4.1. *Let T be a ternary semigroup with zero and I be a 0-essential ideal of T. If I' is an ideal of T containing I, then I' is also a 0-essential ideal of T.*

Proof: Assume that I is a 0-essential ideal of T and let I' be any ideal of T such that $I \subseteq I'$. Let J be any nonzero ideal of T. Thus, $I \cap J \neq \{0\}$. So $I' \cap J \neq \{0\}$ because $I \cap J \subseteq I' \cap J$. Hence, I' is a 0-essential ideal of T. □

Proposition 4.2. *Let T be a ternary semigroup with zero. The following statements hold.*

(1) *The union of any two 0-essential ideals of T is also a 0-essential ideal of T.*
(2) *The intersection of any two 0-essential ideals of T is also a 0-essential ideal of T.*

Proof: (1) This follows by Proposition 4.1.

(2) Assume that I_1 and I_2 are any two 0-essential ideals of T. Then $I_1 \cap I_2$ is an ideal of T and $I_1 \cap I_2 \neq \{0\}$. Let J be any nonzero ideal of T. Then $I_1 \cap J \neq \{0\}$. So there exists a nonzero element $x \in I_1 \cap J$. Let $(x)_i$ be an ideal of T generated by x. Then $(x)_i \neq \{0\}$. So $(x)_i \cap I_2 \neq \{0\}$; thus, there exists a nonzero element $y \in (x)_i \cap I_2$. Then $y \in (I_1 \cap I_2) \cap J$. Hence, $I_1 \cap I_2$ is a 0-essential ideal of T. □

Definition 4.2. *Let T be a ternary semigroup with zero. A fuzzy ideal f of T is called a* ***nontrivial fuzzy ideal*** *of T if there exists a nonzero element $x \in T$ such that $f(x) \neq 0$.*

Next, we introduce the definition of 0-essential fuzzy ideals of a ternary semigroup with zero and provide relationships between those ideals and 0-essential ideals of such ternary semigroup with zero as shown in the following results.

Definition 4.3. *Let T be a ternary semigroup with zero. A fuzzy ideal f of T is called a* ***0-essential fuzzy ideal*** *of T if $supp(f \cap g) \neq \{0\}$ for every nontrivial fuzzy ideal g of T.*

Theorem 4.1. *Let T be a ternary semigroup with zero and I be a nonzero ideal of T. Then I is a 0-essential ideal of T if and only if χ_I is a 0-essential fuzzy ideal of T.*

Proof: Assume that I is a 0-essential ideal of T. Let g be a nontrivial fuzzy ideal of T. Thus, $supp(g)$ is a nonzero ideal of T. So $I \cap supp(g) \neq \{0\}$. Thus, there exists a nonzero element $x \in I \cap supp(g)$, and this implies that $(\chi_I \cap g)(x) \neq 0$. Hence, $x \in supp(\chi_I \cap g)$. Therefore, χ_I is a 0-essential fuzzy ideal of T. To prove the converse, assume that χ_I is a 0-essential fuzzy ideal of T and let J be a nonzero ideal of T. Thus, χ_J is a nontrivial fuzzy ideal of T. So $supp(\chi_I \cap \chi_J) \neq \{0\}$. Thus, $\chi_{I \cap J} \neq \chi_{\{0\}}$. Hence, $I \cap J \neq \{0\}$. □

Theorem 4.2. *Let T be a ternary semigroup with zero and f be any fuzzy ideal of T. Then f is a 0-essential fuzzy ideal of T if and only if $supp(f)$ is a 0-essential ideal of T.*

Proof: Assume that I is a 0-essential ideal of T and let g be a nontrivial fuzzy ideal of T. Then $supp(g)$ is a nonzero ideal of T. So $I \cap supp(g) \neq \{0\}$. Thus, there exists a nonzero element $x \in I \cap supp(g)$. It implies that $(\chi_I \cap g)(x) \neq 0$. Hence, $supp(\chi_I \cap g) \neq \{0\}$. Therefore, χ_I is a 0-essential fuzzy ideal of T. To prove the converse, assume that χ_I is a 0-essential fuzzy ideal of T and let J be a nonzero ideal of T. Thus, χ_J is a nontrivial fuzzy ideal of T. Then $supp(\chi_I \cap \chi_J) \neq \{0\}$. Therefore, $I \cap J \neq \{0\}$. □

Proposition 4.3. *Let T be a ternary semigroup with zero and f be a* 0*-essential fuzzy ideal of T. If f' is a fuzzy ideal of T such that $f \subseteq f'$, then f' is also a* 0*-essential fuzzy ideal of T.*

Proof: Assume that f is a 0-essential fuzzy ideal of T and let f' be a fuzzy ideal of T such that $f \subseteq f'$. Let g be any nontrivial fuzzy ideal of T. Thus, $supp(f \cap g) \neq \{0\}$. This implies that $supp(f' \cap g) \neq \{0\}$. Hence, f' is a 0-essential fuzzy ideal of T. □

Theorem 4.3. *Let T be a ternary semigroup with zero. The following statements are true.*

(1) *The union of any two* 0*-essential fuzzy ideals of T is a* 0*-essential fuzzy ideal of T.*
(2) *The intersection of any two* 0*-essential fuzzy ideals of T is a* 0*-essential fuzzy ideal of T.*

Proof: (1) It follows by Proposition 4.3.

(2) Assume that f_1 and f_2 are any two 0-essential fuzzy ideals of T. Thus, $f_1 \cap f_2$ is a fuzzy ideal of T. Let g be any nontrivial fuzzy ideal of T. This implies that $supp(f_1 \cap g) \neq \{0\}$. So there exists a nonzero element $x \in T$ such that $(f_1 \cap g)(x) \neq 0$. Since f_2 is a 0-essential fuzzy ideal of T, we have $supp(f_2)$ is a 0-essential ideal of T. Hence, $supp(f_2) \cap (x)_i \neq \{0\}$. Then there exists a nonzero element $y \in supp(f_2) \cap (x)_i$. This implies that $f_2(y) \neq 0$. Since f_1 and g are fuzzy ideals of T, $f_1(y) \geq f_1(x)$ and $g(y) \geq g(x)$. So $((f_1 \cap f_2) \cap g)(y) \neq 0$. Then $supp[(f_1 \cap f_2) \cap g] \neq \{0\}$. Hence, $f_1 \cap f_2$ is a 0-essential fuzzy ideal of T. □

Next, we extend the concept of minimal (prime, semiprime) essential ideals and minimal (prime, semiprime) essential fuzzy ideals of a ternary semigroup in Section 3 to the concept of minimal (prime, semiprime) 0-essential ideals and minimal (prime, semiprime) 0-essential fuzzy ideals of a ternary semigroup with zero, respectively.

Definition 4.4. *Let T be a ternary semigroup with zero. A* 0*-essential ideal I of T is called* ***minimal*** *if for every* 0*-essential ideal J of T such that $J \subseteq I$, we have $J = I$.*

Example 4.2. *Consider the ternary semigroup $(\mathbb{Z}_{12}, +)$. We have that $\{0, 2, 4, 6, 8, 10\}$ is a unique* 0*-essential ideals of $\mathbb{Z}_{12}$.*

Definition 4.5. *Let T be a ternary semigroup with zero. A* 0*-essential fuzzy ideal f of T is called* ***minimal*** *if for every* 0*-essential fuzzy ideal g of T such that $g \subseteq f$, we have $supp(f) = supp(g)$.*

Theorem 4.4. *Let T be a ternary semigroup with zero and A be a nonempty subset of T. Thus, A is a minimal* 0*-essential ideal of T if and only if χ_A is a minimal* 0*-essential fuzzy ideal of T.*

Proof: This is similar to Theorem 3.4. □

Definition 4.6. *Let T be a ternary semigroup with zero.*

(1) *A* 0*-essential ideal A of T is called* ***prime*** *if for all $x, y, z \in T$, $[xyz] \in A$ implies $x \in A$ or $y \in A$ or $z \in A$.*
(2) *A* 0*-essential fuzzy ideal g of T is called* ***prime*** *if for all $x, y, z \in T$, $f([xyz]) \leq \max\{f(x), f(y), f(z)\}$.*

The next theorem shows the relationship between prime 0-essential ideals and prime 0-essential fuzzy ideals of a ternary semigroup with zero.

Theorem 4.5. *Let T be a ternary semigroup with zero and A be a nonempty subset of T. Then A is a prime* 0*-essential ideal of T if and only if χ_A is a prime* 0*-essential fuzzy ideal of T.*

Proof: This is similar to Theorem 3.5. □

Definition 4.7. *Let T be a ternary semigroup with zero.*

(1) *A 0-essential ideal A of T is called **semiprime** if for all $x \in T$, $[xxx] \in A$ implies $x \in A$.*
(2) *A 0-essential fuzzy ideal f of T is called **semiprime** if for all $x \in T$, $f([xxx]) \leq f(x)$.*

Finally, we give the relationship between semiprime 0-essential ideals and semiprime 0-essential fuzzy ideals of a ternary semigroup with zero.

Theorem 4.6. *Let T be a ternary semigroup with zero and A be a nonempty subset of T. Then A is a semiprime 0-essential ideal of T if and only if χ_A is a semiprime 0-essential fuzzy ideal of T.*

Proof: This is similar to Theorem 3.6. □

5. **Conclusion.** In this paper, we proved that the union and intersection of essential ideals (0-essential ideals, essential fuzzy ideals, 0-essential fuzzy ideals) of a ternary semigroup are also an essential ideal (0-essential ideals, essential fuzzy ideals, 0-essential fuzzy ideals) of that ternary semigroup, respectively.

In addition, the study revealed that an ideal I of a ternary semigroup T is essential if and only if χ_I is an essential fuzzy ideal of T and it was found that a nonzero fuzzy ideal f of T is an essential fuzzy ideal if and only if $supp(f)$ is essential. In the same manner, the study indicated that a nonzero ideal I of a ternary semigroup with zero T is 0-essential if and only if χ_I is a 0-essential fuzzy ideal of T and it was found that any fuzzy ideal f of T is a 0-essential fuzzy ideal if and only if $supp(f)$ is 0-essential.

Moreover, we showed that a nonempty subset A of a ternary semigroup T is a minimal (prime, semiprime) essential ideal of T if and only if χ_A is a minimal (prime, semiprime) essential fuzzy ideal of T, respectively. In the same way, we obtained that a nonempty subset A of a ternary semigroup with zero T is a minimal (prime, semiprime) 0-essential ideal of T if and only if χ_A is a minimal (prime, semiprime) 0-essential fuzzy ideal of T, respectively.

In the future work, we can study essential ideals and their fuzzifications in another algebraic structures. For example, we can define essential hyperideals and essential fuzzy hyperideals in semihypergroups.

REFERENCES

[1] D. H. Lehmer, A ternary analogue of Abelian groups, *American Journal of Mathematics*, vol.54, pp.329-338, 1932.
[2] J. Los, On the extending of models (I), *Fundamenta Mathematicae*, vol.42, pp.38-54, 1955.
[3] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol.8, no.3, pp.338-353, 1965.
[4] M. A. Ansari and N. Yaqoob, T-rough ideals in ternary semigroups, *International Journal of Pure and Applied Mathematics*, vol.86, pp.411-424, 2013.
[5] R. Chinram, S. Kaewchay, A. Iampan and P. Singavananda, Characterization of almost ternary subsemigroups and their fuzzifications, *Journal of Mathematics and Computer Science*, vol.27, no.2, pp.97-104, 2022.
[6] P. Julatha and A. Iampan, A new generalization of hesitant and interval-valued fuzzy ideals of ternary semigroups, *International Journal of Fuzzy Logic and Intelligent Systems*, vol.21, no.2, pp.169-175, 2021.
[7] N. Yaqoob and M. A. Ansari, Bipolar (λ, δ)-fuzzy ideals in ternary semigroups, *International Journal of Mathematical Analysis*, vol.7, pp.1775-1782, 2013.
[8] U. Medhi, K. Rajkhowa, L. K. Barthakur and H. K. Saikia, On fuzzy essential ideals of ring, *Advances in Fuzzy Sets and Systems*, vol.3, pp.287-299, 2008.
[9] U. Medhi and H. K. Saikia, On T-fuzzy essential ideals of rings, *International Journal of Pure and Applied Mathematics*, vol.89, pp.343-353, 2013.

[10] S. Baupradist, B. Chemat, K. Palanivel and R. Chinram, Essential ideals and essential fuzzy ideals in semigroups, *Journal of Discrete Mathematical Sciences and Cryptography*, vol.24, no.1, pp.223-233, 2021.
[11] R. Chinram and T. Gaketem, Essential (m, n)-ideal and essential fuzzy (m, n)-ideals in semigroups, *ICIC Express Letters*, vol.15, no.10, pp.1037-1044, 2021.
[12] T. Gaketem and A. Iampan, Essential UP-ideals and t-essential fuzzy UP-ideals of UP-algebras, *ICIC Express Letters*, vol.15. no.12, pp.1283-1289, 2021.