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ON LEFT WEAK-INTERIOR IDEALS AND RIGHT WEAK-INTERIOR BASES OF ORDERED SEMIGROUPS

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Abstract. *Bases of ideals is one of important researches in algebraic structures. We introduce the concept of left weak-interior ideals of ordered semigroups and investigate their properties. The main result of this paper is to introduce the concept of right weak-interior bases of an ordered semigroup which is based on the results of left weak-interior ideals generated by a non-empty subset of an ordered semigroup. Moreover, we focus on generalizing the results of right weak-interior bases of semigroups to ordered semigroups.*
Keywords: Ordered semigroup, Left weak-interior ideal, Right weak-interior base, Quasi-order

1. **Introduction.** Based on the concept of one-sided ideals of a semigroup generated by a non-empty set, the concept of one-sided bases of a semigroup has been introduced and studied by Fabrici [1]. Later, the concept of bases of another ideals of semigroups was studied. Panate et al. [2] introduced the concept of interior bases of semigroups. Likewise, Jantanan et al. [3] studied and discussed the concept of interior bases of semigroups. Recently, Jantanan et al. [4] generalized the results in [3] to study left and right weak-interior bases of a semigroup. Now, the concept of one-sided bases has been studied in other algebraic structures [5, 6, 7, 8, 9]. An ordered semigroup is one of generalizations of semigroups. Many basic notations and remarkable results of semigroups were extended to notations and results of ordered semigroups. This is the motivation of this paper. The main purpose of this paper is to generalize the results obtained in [4] from semigroups to ordered semigroups. We define the concept of left weak-interior ideals of an ordered semigroup and investigate their properties. Moreover, we introduce the concept of right weak-interior bases of an ordered semigroup which is based on the results of left weak-interior ideals generated by a non-empty subset of an ordered semigroup and provide some properties of right weak-interior bases of an ordered semigroup. In Section 2, we recall

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some basic definitions and results of ordered semigroups. In Section 3, we introduce left weak-interior ideals in ordered semigroups and investigate their basic results. Section 4 is the main part of this paper; we show remarkable results of left weak-interior bases of ordered semigroups. Moreover, we note that some results in [4] are special cases of results of this paper. Finally, Section 5 concludes the paper.

2. **Preliminaries.** The structure $(S, \cdot, \leq)$ is called an *ordered semigroup* if $(S, \cdot)$ is a semigroup, $(S, \leq)$ is a partially ordered set and for any $a, b, c \in S$, $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. For simplicity, we will denote an ordered semigroup $(S, \cdot, \leq)$ by S, and if $a, b \in S$, we will replace $a \cdot b$ by ab. For any two non-empty subsets A and B of an ordered semigroup S, the *product* of A and B is defined by $AB := \{ab \mid a \in A \text{ and } b \in B\}$. For $a \in S$, we will replace $\{a\}B$ by aB, and for $H \subseteq S$, we denote

$$(H] := \{a \in S \mid a \leq h \text{ for some } h \in H\}.$$

Let S be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if for any $a, b \in A$, $ab \in A$ or $AA \subseteq A$. A non-empty subset A of S is called a *left* (resp. *right*) *ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$) and $A = (A]$, that is, for any $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.

Proposition 2.1. [10] *Let S be an ordered semigroup and A, B be any two non-empty subsets of S. Then the following statements hold.*

(1) $A \subseteq (A]$ *and* $((A]] = (A]$.
(2) *If* $A \subseteq B$, *then* $(A] \subseteq (B]$.
(3) $(A](B] \subseteq (AB]$.
(4) $(A \cup B] = (A] \cup (B]$.

Proposition 2.2. [10] *Let S be an ordered semigroup and $\{A_i \mid i \in I\}$ be a non-empty family of subsemigroups of S. Then $\bigcap_{i \in I} A_i = \varnothing$ or $\bigcap_{i \in I} A_i$ is a subsemigroup of S.*

3. **Left Weak-Interior Ideals in Ordered Semigroups.** We begin this section with the definition of left weak-interior ideals of an ordered semigroup as follows.

Definition 3.1. *Let S be an ordered semigroup. A non-empty subset A of S is called a* left weak-interior ideal *of S if it satisfies*

(1) *A is a subsemigroup of S;*
(2) $SAA \subseteq A$;
(3) $A = (A]$, *that is, for any $x \in A$ and $y \in S$, $y \leq x$ implies $y \in A$.*

The concept of right weak-interior ideals of an ordered semigroup can be defined dually. In the next example, we consider an ordered semigroup from [11].

Example 3.1. *Consider the ordered semigroup $S = \{a, b, c, d\}$ under the binary operation $\cdot$ and the partial order relation $\leq$ below:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

Then the left weak-interior ideals of S are $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$ and S. It is observed that $\{a, c\}$ is a left weak-interior ideal of S, but it is not a left ideal of S. Thus, we infer that in general, a left weak-interior ideal of S need not be a left ideal of S.

Proposition 3.1. *Let S be an ordered semigroup and $\{A_i \mid i \in I\}$ be a non-empty family of left weak-interior ideals of S. Then $\bigcap_{i\in I} A_i = \varnothing$ or $\bigcap_{i\in I} A_i$ is a left weak-interior ideal of S.*

Proof: Suppose that $\bigcap_{i\in I} A_i \neq \varnothing$. By Proposition 2.2, $\bigcap_{i\in I} A_i$ is a subsemigroup of S. Next, for all $i \in I$, we obtain $S\left(\bigcap_{i\in I} A_i\right)\left(\bigcap_{i\in I} A_i\right) \subseteq SA_iA_i \subseteq A_i$. Thus, $S\left(\bigcap_{i\in I} A_i\right)\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} A_i$ and $\left(\bigcap_{i\in I} A_i\right] \subseteq \bigcap_{i\in I}(A_i] = \bigcap_{i\in I} A_i$. Therefore, $\bigcap_{i\in I} A_i$ is a left weak-interior ideal of S. □

Let S be an ordered semigroup. Then the intersection of all left weak-interior ideals of S containing a non-empty subset A of S, denoted by $L_{wi}(A)$ as the smallest left weak-interior ideal of S containing A. For any $a \in S$, we denote $L_{wi}(a) = L_{wi}(\{a\})$. The form of $L_{wi}(A)$ is shown in the following proposition.

Proposition 3.2. *Let S be an ordered semigroup and A be a non-empty subset of S. Then*

$$L_{wi}(A) = (A \cup AA \cup SAA].$$

Proof: Let $B = (A \cup AA \cup SAA]$. Then, we have

$$\begin{aligned} BB &= (A \cup AA \cup SAA](A \cup AA \cup SAA] \subseteq ((A \cup AA \cup SAA)(A \cup AA \cup SAA)] \\ &\subseteq (AA \cup SAA] \subseteq B, \\ SBB &= S(A \cup AA \cup SAA](A \cup AA \cup SAA] \subseteq (S]((A \cup AA \cup SAA)(A \cup AA \cup SAA)] \\ &\subseteq (S(AA \cup SAA)] \subseteq (SAA] \subseteq B \end{aligned}$$

and $(B] = ((A \cup AA \cup SAA]] = (A \cup AA \cup SAA] = B$. Hence, B is a left weak-interior ideal of S. Next, let B' be a left weak-interior ideal of S containing A. Since $A \subseteq B'$, we have $AA \subseteq B'B' \subseteq B'$ and $SAA \subseteq SB'B' \subseteq B'$. Thus, $A \cup AA \cup SAA \subseteq B'$. It follows that $B = (A \cup AA \cup SAA] \subseteq (B'] = B'$. Hence, B is the smallest left weak-interior ideal of S containing A. This shows that $L_{wi}(A) = (A \cup AA \cup SAA]$. □

Corollary 3.1. *If a is an element of an ordered semigroup S, then $L_{wi}(a) = (a \cup aa \cup Saa]$.*

4. **Right Weak-Interior Bases in Ordered Semigroups.** In this section, we define the concept of right weak-interior bases of an ordered semigroup and study the structure of an ordered semigroup containing right weak-interior bases. It is observed that the results for an ordered semigroup containing left weak-interior bases are left-right dual.

Definition 4.1. *Let S be an ordered semigroup. A non-empty subset A of S is called a* right weak-interior base *of S if it satisfies the following two conditions:*

(1) *$S = L_{wi}(A)$;*
(2) *if B is a subset of A such that $S = L_{wi}(B)$, then $B = A$.*

In the following example, we consider an ordered semigroup from [12].

Example 4.1. *Consider the ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the partial order relation $\leq$ below:*

$\cdot$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (c, e)\}.$$

Then the right weak-interior bases of S are $\{a, e\}$ and $\{b, e\}$. However, $\{a, b, e\}$ is not a right weak-interior base of S.

In the next example, we consider an ordered semigroup from [11].

Example 4.2. *Consider the ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the partial order relation $\leq$ below:*

$\cdot$	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, d), (c, e)\}.$$

Then the right weak-interior base of S is $\{b, d\}$. However, both $\{b\}$ and $\{d\}$ are not right weak-interior bases of S.

Lemma 4.1. *Let A be a right weak-interior base of an ordered semigroup S, and $a, b \in A$. If $a \in (bb \cup Sbb]$, then $a = b$.*

Proof: Assume that $a \in (bb \cup Sbb]$, and suppose that $a \neq b$. We consider $B = A \setminus \{a\}$. Clearly, $B \subset A$. Since $a \neq b$, then $b \in B$. First, to show that $L_{wi}(A) \subseteq L_{wi}(B)$, we let $x \in L_{wi}(A)$. Then $x \in (A \cup AA \cup SAA]$. So, we have $x \leq y$ for some $y \in A \cup AA \cup SAA$. There are three cases to consider.

Case 1: $y \in A$.

Subcase 1.1: $y \neq a$. It follows that $y \in A \setminus \{a\} = B \subseteq L_{wi}(B)$.

Subcase 1.2: $y = a$. Since $a \in (bb \cup Sbb]$, it follows that

$$y = a \in (bb \cup Sbb] \subseteq (BB \cup SBB] \subseteq L_{wi}(B).$$

Case 2: $y \in AA$. Then $y = b_1 b_2$ for some $b_1, b_2 \in A$.

Subcase 2.1: $b_1 \neq a$ and $b_2 \neq a$. Since $B = A \setminus \{a\}$, it follows that

$$y = b_1 b_2 \in (A \setminus \{a\})(A \setminus \{a\}) = BB \subseteq L_{wi}(B).$$

Subcase 2.2: $b_1 \neq a$ and $b_2 = a$. Since $B = A \setminus \{a\}$ and $a \in (bb \cup Sbb]$, it follows that

$$\begin{aligned} y = b_1 b_2 &\in B(bb \cup Sbb] \subseteq (B](BB \cup SBB] \subseteq ((B)(BB \cup SBB)] \\ &= (BBB \cup BSBB] \subseteq (SBB] \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.3: $b_1 = a$ and $b_2 \neq a$. Since $a \in (bb \cup Sbb]$ and $B = A \setminus \{a\}$, it follows that

$$\begin{aligned} y = b_1 b_2 &\in (bb \cup Sbb]B \subseteq (BB \cup SBB](B] \subseteq ((BB \cup SBB)(B)] \\ &= (BBB \cup SBBB] \subseteq (SBB] \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.4: $b_1 = a$ and $b_2 = a$. Since $a \in (bb \cup Sbb]$, it follows that

$$\begin{aligned} y = b_1 b_2 &\in (bb \cup Sbb](bb \cup Sbb] \subseteq ((bb \cup Sbb)(bb \cup Sbb)] \\ &= (bbbb \cup bbSbb \cup Sbbbb \cup SbbSbb] \\ &\subseteq (BBBB \cup BBSBB \cup SBBBB \cup SBBSBB] \\ &\subseteq (SBB] \subseteq L_{wi}(B). \end{aligned}$$

Case 3: $y \in SAA$. Then $y = sb_3 b_4$ for some $s \in S$ and $b_3, b_4 \in A$.

Subcase 3.1: $b_3 \neq a$ and $b_4 \neq a$. Since $B = A \setminus \{a\}$, it follows that

$$y = sb_3 b_4 \in S(A \setminus \{a\})(A \setminus \{a\}) = SBB \subseteq L_{wi}(B).$$

Subcase 3.2: $b_3 \neq a$ and $b_4 = a$. Since $B = A \setminus \{a\}$ and $a \in (bb \cup Sbb]$, it follows that

$$\begin{aligned} y = sb_3 b_4 &\in SB(bb \cup Sbb] \subseteq (S](BB \cup SBB] \subseteq ((S)(BB \cup SBB)] \\ &= (SBB \cup SSBB] \subseteq (SBB] \subseteq L_{wi}(B). \end{aligned}$$

Subcase 3.3: $b_3 = a$ and $b_4 \neq a$. Since $a \in (bb \cup Sbb]$ and $B = A \setminus \{a\}$, it follows that

$$\begin{aligned} y = sb_3b_4 &\in S(bb \cup Sbb]B \subseteq (S](BB \cup SBB](B] \\ &\subseteq ((S)(BB \cup SBB)](B] \subseteq ((SBB \cup SSBB)(B)] \\ &= (SBBB \cup SSBBB] \subseteq (SBB] \subseteq L_{wi}(B). \end{aligned}$$

Subcase 3.4: $b_3 = a$ and $b_4 = a$. Since $a \in (bb \cup Sbb]$, it follows that

$$\begin{aligned} y = sb_3b_4 &\in S(bb \cup Sbb](bb \cup Sbb] \\ &\subseteq (S]((BB \cup SBB)(BB \cup SBB)] \\ &\subseteq ((S)(BBBB \cup BBSBB \cup SBBBB \cup SBBSBB)] \\ &= (SBBBB \cup SBBSBB \cup SSBBBB \cup SSBBSBB] \\ &\subseteq (SBB] \subseteq L_{wi}(B). \end{aligned}$$

In all these cases, we have that $y \in L_{wi}(B)$. Since $x \leq y$ and $y \in L_{wi}(B)$, it follows that $x \in (L_{wi}(B)] = L_{wi}(B)$. Thus, $L_{wi}(A) \subseteq L_{wi}(B)$. Since A is a right weak-interior base of S, it follows that $S = L_{wi}(A) \subseteq L_{wi}(B) \subseteq S$. Hence, $L_{wi}(B) = S$. This is a contradiction. Therefore, $a = b$. □

Lemma 4.2. *Let A be a right weak-interior base of an ordered semigroup S, and $a, b, c \in A$. If $a \in (bc \cup Sbc]$, then $a = b$ or $a = c$.*

Proof: Assume that $a \in (bc \cup Sbc]$, and suppose that $a \neq b$ and $a \neq c$. We consider $B = A \setminus \{a\}$. Clearly, $B \subset A$. Since $a \neq b$ and $a \neq c$, then $b, c \in B$. We will show that $L_{wi}(A) \subseteq L_{wi}(B)$, it suffices to show that $A \subseteq L_{wi}(B)$. Let $x \in A$. If $x \neq a$, then $x \in A \setminus \{a\} = B \subseteq L_{wi}(B)$. If $x = a$, then $x = a \in (bc \cup Sbc] \subseteq (BB \cup SBB] \subseteq L_{wi}(B)$. Thus, $A \subseteq L_{wi}(B)$. Since $A \subseteq L_{wi}(B)$ and $L_{wi}(B)$ is a left weak-interior ideal of S, we obtain that $AA \subseteq L_{wi}(B)L_{wi}(B) \subseteq L_{wi}(B)$ and $SAA \subseteq SL_{wi}(B)L_{wi}(B) \subseteq L_{wi}(B)$. Hence, $A \cup AA \cup SAA \subseteq L_{wi}(B)$, and so $L_{wi}(A) = (A \cup AA \cup SAA] \subseteq (L_{wi}(B)] = L_{wi}(B)$. This implies that $L_{wi}(A) \subseteq L_{wi}(B)$. Since A is a right weak-interior base of S, it follows that $S = L_{wi}(A) \subseteq L_{wi}(B) \subseteq S$. Hence, $L_{wi}(B) = S$. This is a contradiction. Therefore, $a = b$ or $a = c$. □

Besides the partial order $\leq$ on an ordered semigroup S, we need the quasi-order defined as follows.

Definition 4.2. *Let S be an ordered semigroup. Define a* quasi-order *on S by, for any $a, b \in S$, $a \leq_{Lwi} b$ if $L_{wi}(a) \subseteq L_{wi}(b)$.*

It is clear that, for any $a, b \in S$, $a \leq b$ implies $a \leq_{Lwi} b$. The following example shows that the converse statement is not valid in general.

Example 4.3. *From Example 4.2, we obtain that $c \leq_{Lwi} b$, but $c \leq b$ is false.*

Lemma 4.3. *If A is a right weak-interior base of an ordered semigroup S, then for any $x \in S$ there exists $a \in A$ such that $L_{wi}(x) \subseteq L_{wi}(a)$.*

Proof: Suppose that A is a right weak-interior base of S. We have $L_{wi}(A) = S$. Let $x \in S$. Then $x \in L_{wi}(A)$, and so $x \in L_{wi}(a)$ for some $a \in A$. Since $x \in L_{wi}(a)$ and $L_{wi}(a)$ is a left weak-interior ideal of S, we obtain that $xx \subseteq L_{wi}(a)L_{wi}(a) \subseteq L_{wi}(a)$ and $Sxx \subseteq SL_{wi}(a)L_{wi}(a) \subseteq L_{wi}(a)$. It follows that $x \cup xx \cup Sxx \subseteq L_{wi}(a)$, and so $L_{wi}(x) = (x \cup xx \cup Sxx] \subseteq (L_{wi}(a)] = L_{wi}(a)$. Hence, $L_{wi}(x) \subseteq L_{wi}(a)$. □

Lemma 4.4. *Let A be a right weak-interior base of an ordered semigroup S. If $a, b \in A$ such that $a \neq b$, then neither $a \leq_{Lwi} b$ nor $b \leq_{Lwi} a$.*

Proof: Assume that $a, b \in A$ such that $a \neq b$. Suppose that $a \leq_{Lwi} b$. Then $L_{wi}(a) \subseteq L_{wi}(b)$. We consider $B = A \setminus \{a\}$. Clearly, $B \subset A$. Since $a \neq b$, then $b \in B$. Now, show that $L_{wi}(B) = S$. Clearly, $L_{wi}(B) \subseteq S$. Let $x \in S$. By Lemma 4.3, there exists $a_1 \in A$ such that $L_{wi}(x) \subseteq L_{wi}(a_1)$. If $a_1 \neq a$, we have $a_1 \in B$, and so $x \in L_{wi}(x) \subseteq L_{wi}(a_1) \subseteq L_{wi}(B)$. If $a_1 = a$, then $L_{wi}(x) \subseteq L_{wi}(a_1) = L_{wi}(a)$, and so $x \in L_{wi}(x) \subseteq L_{wi}(a) \subseteq L_{wi}(b) \subseteq L_{wi}(B)$. Thus, $S \subseteq L_{wi}(B)$. Hence, $L_{wi}(B) = S$. This is a contradiction. This shows that $a \leq_{Lwi} b$ is false. For the case $b \leq_{Lwi} a$, we can be proved similarly. □

Lemma 4.5. *Let A be a right weak-interior base of an ordered semigroup S. For any $a, b, c \in A$ and $s \in S$, then the following statements hold:*

(1) *if $a \in (bc \cup bcbc \cup Sbcbc]$, then $a = b$ or $a = c$.*
(2) *if $a \in (sbc \cup sbcsbc \cup Ssbcsbc]$, then $a = b$ or $a = c$.*

Proof: (1) Assume that $a \in (bc \cup bcbc \cup Sbcbc]$. Suppose that $a \neq b$ and $a \neq c$. We set $B = A \setminus \{a\}$. Clearly, $B \subset A$. Since $a \neq b$ and $a \neq c$, then $b, c \in B$. We will show that $L_{wi}(A) \subseteq L_{wi}(B)$. It is sufficient to prove that $A \subseteq L_{wi}(B)$. Let $x \in A$. If $x \neq a$, then $x \in B$. Thus, $x \in B \subseteq L_{wi}(B)$. If $x = a$, then

$$x = a \in (bc \cup bcbc \cup Sbcbc] \subseteq (BB \cup BBBB \cup SBBBB] \subseteq (BB \cup SBB] \subseteq L_{wi}(B).$$

Hence, $x \in L_{wi}(B)$, and so $A \subseteq L_{wi}(B)$. Since $A \subseteq L_{wi}(B)$ and $L_{wi}(B)$ is a left weak-interior ideal of S, we have that

$$AA \subseteq L_{wi}(B)L_{wi}(B) \subseteq L_{wi}(B) \text{ and } SAA \subseteq SL_{wi}(B)L_{wi}(B) \subseteq L_{wi}(B).$$

Thus, $A \cup AA \cup SAA \subseteq L_{wi}(B)$, and so $L_{wi}(A) = (A \cup AA \cup SAA] \subseteq (L_{wi}(B)] = L_{wi}(B)$. Since A is a right weak-interior base of S, it follows that $S = L_{wi}(A) \subseteq L_{wi}(B) \subseteq S$. This implies that $L_{wi}(B) = S$. This is a contradiction.

(2) Assume that $a \in (sbc \cup sbcsbc \cup Ssbcsbc]$. Suppose that $a \neq b$ and $a \neq c$. Setting $B = A \setminus \{a\}$. Then $B \subset A$. Since $a \neq b$ and $a \neq c$, we have $b, c \in B$. We claim that $A \subseteq L_{wi}(B)$. Let $x \in A$. If $x \neq a$, then $x \in B \subseteq L_{wi}(B)$. If $x = a$, then

$$x = a \in (sbc \cup sbcsbc \cup Ssbcsbc] \subseteq (SBB \cup SBBSBB \cup SSBBSBB] \subseteq (SBB] \subseteq L_{wi}(B).$$

Thus, $x \in L_{wi}(B)$, and so $A \subseteq L_{wi}(B)$. Since $A \subseteq L_{wi}(B)$ and $L_{wi}(B)$ is a left weak-interior ideal of S, we obtain that

$$AA \subseteq L_{wi}(B)L_{wi}(B) \subseteq L_{wi}(B) \text{ and } SAA \subseteq SL_{wi}(B)L_{wi}(B) \subseteq L_{wi}(B).$$

Hence, $A \cup AA \cup SAA \subseteq L_{wi}(B)$, and so $L_{wi}(A) = (A \cup AA \cup SAA] \subseteq (L_{wi}(B)] = L_{wi}(B)$. Since A is a right weak-interior base of S, it follows that $S = L_{wi}(A) \subseteq L_{wi}(B) \subseteq S$. Thus, $L_{wi}(B) = S$. This is a contradiction. □

Lemma 4.6. *Let A be a right weak-interior base of an ordered semigroup S. Then the following statements hold:*

(1) *for any $a, b, c \in A$, if $a \neq b$ and $a \neq c$, then $a \nleq_{Lwi} bc$;*
(2) *for any $a, b, c \in A$ and $s \in S$, if $a \neq b$ and $a \neq c$, then $a \nleq_{Lwi} sbc$.*

Proof: (1) Let $a, b, c \in A$ such that $a \neq b$ and $a \neq c$. Suppose that $a \leq_{Lwi} bc$. Then

$$a \in L_{wi}(a) \subseteq L_{wi}(bc) = (bc \cup bcbc \cup Sbcbc].$$

By Lemma 4.5(1), we have that $a = b$ or $a = c$, which is a contradiction. Thus, $a \nleq_{Lwi} bc$.

(2) Let $a, b, c \in A$ and $s \in S$ such that $a \neq b$ and $a \neq c$. Suppose that $a \leq_{Lwi} sbc$, then

$$a \in L_{wi}(a) \subseteq L_{wi}(sbc) = (sbc \cup sbcsbc \cup Ssbcsbc].$$

By Lemma 4.5(2), we obtain that $a = b$ or $a = c$. This is a contradiction. Thus, $a \nleq_{Lwi} sbc$. □

The following theorem characterizes when a non-empty subset of an ordered semigroup S is a right weak-interior base of S.

Theorem 4.1. *A non-empty subset A of an ordered semigroup S is a right weak-interior base of S if and only if A satisfies the following conditions:*

(1) *for any* $x \in S$,
  (1.1) *there exists* $a \in A$ *such that* $x \leq_{Lwi} a$; *or*
  (1.2) *there exist* $a_1, a_2 \in A$ *such that* $x \leq_{Lwi} a_1a_2$; *or*
  (1.3) *there exist* $a_3, a_4 \in A$, $s \in S$ *such that* $x \leq_{Lwi} sa_3a_4$;
(2) *for any* $a, b, c \in A$, *if* $a \neq b$ *and* $a \neq c$, *then* $a \nleq_{Lwi} bc$;
(3) *for any* $a, b, c \in A$ *and* $s \in S$, *if* $a \neq b$ *and* $a \neq c$, *then* $a \nleq_{Lwi} sbc$.

Proof: Assume that A is a right weak-interior base of S. First, to show that (1) holds, let $x \in S$. By assumption, $S = L_{wi}(A)$. Thus, $x \in L_{wi}(A) = (A \cup AA \cup SAA]$, and so $x \leq y$ for some $y \in A \cup AA \cup SAA$. There are three cases to consider:
Case 1: $y \in A$. Then $y = a$ for some $a \in A$. Since $a \in A$ and A is a right weak-interior base of S. By Lemma 4.3, we obtain $L_{wi}(x) \subseteq L_{wi}(a)$. Hence, $x \leq_{Lwi} a$.
Case 2: $y \in AA$. Then $y = a_1a_2$ for some $a_1, a_2 \in A$. It follows that $L_{wi}(y) = L_{wi}(a_1a_2)$. Since $x \leq y$ and $y \in L_{wi}(a_1a_2)$, we have $x \in (L_{wi}(a_1a_2)] = L_{wi}(a_1a_2)$. Since $x \in L_{wi}(a_1a_2)$ and $L_{wi}(a_1a_2)$ is a left weak-interior ideal of S, then

$$xx \subseteq L_{wi}(a_1a_2)L_{wi}(a_1a_2) \subseteq L_{wi}(a_1a_2) \text{ and } Sxx \subseteq SL_{wi}(a_1a_2)L_{wi}(a_1a_2) \subseteq L_{wi}(a_1a_2).$$

Thus, $x \cup xx \cup Sxx \subseteq L_{wi}(a_1a_2)$, and so $L_{wi}(x) = (x \cup xx \cup Sxx] \subseteq (L_{wi}(a_1a_2)] = L_{wi}(a_1a_2)$. This implies that $x \leq_{Lwi} a_1a_2$.
Case 3: $y \in SAA$. Then $y = sa_3a_4$ for some $a_3, a_4 \in A$ and $s \in S$. It follows that $L_{wi}(y) = L_{wi}(sa_3a_4)$. Since $x \leq y$ and $y \in L_{wi}(sa_3a_4)$, we have $x \in (L_{wi}(sa_3a_4)] = L_{wi}(sa_3a_4)$. Since $x \in L_{wi}(sa_3a_4)$ and $L_{wi}(sa_3a_4)$ is a left weak-interior ideal of S, we obtain that

$$\begin{aligned} xx &\subseteq L_{wi}(sa_3a_4)L_{wi}(sa_3a_4) \subseteq L_{wi}(sa_3a_4) \\ \text{and } Sxx &\subseteq SL_{wi}(sa_3a_4)L_{wi}(sa_3a_4) \subseteq L_{wi}(sa_3a_4). \end{aligned}$$

Thus, $x \cup xx \cup Sxx \subseteq L_{wi}(sa_3a_4)$, and so $L_{wi}(x) = (x \cup xx \cup Sxx] \subseteq (L_{wi}(sa_3a_4)] = L_{wi}(sa_3a_4)$. This implies that $x \leq_{Lwi} sa_3a_4$.

The conditions (2) and (3) hold from Lemma 4.6(1) and Lemma 4.6(2), respectively.

Conversely, assume that the conditions (1), (2) and (3) hold. We will show that A is a right weak-interior base of S. First, we need to show that $S = L_{wi}(A)$. Clearly, $L_{wi}(A) \subseteq S$. Let $x \in S$. If (1.1) holds, we have $a \in A$ such that $x \leq_{Lwi} a$. Thus, $x \in L_{wi}(x) \subseteq L_{wi}(a) \subseteq L_{wi}(A)$, and so $x \in L_{wi}(A)$. If (1.2) holds, we have $a_1, a_2 \in A$ such that $x \leq_{Lwi} a_1a_2$. Thus, $x \in L_{wi}(x) \subseteq L_{wi}(a_1a_2)$ and

$$\begin{aligned} L_{wi}(a_1a_2) &= (a_1a_2 \cup a_1a_2a_1a_2 \cup Sa_1a_2a_1a_2] \subseteq (AA \cup AAAA \cup SAAAA] \\ &\subseteq (AA \cup SAA] \subseteq L_{wi}(A). \end{aligned}$$

It follows that $x \in L_{wi}(A)$. If (1.3) holds, then there exists $a_3, a_4 \in A$ and $s \in S$, such that $x \leq_{Lwi} sa_3a_4$. So, we obtain that $x \in L_{wi}(x) \subseteq L_{wi}(sa_3a_4)$ and

$$\begin{aligned} L_{wi}(sa_3a_4) &= (sa_3a_4 \cup sa_3a_4sa_3a_4 \cup Ssa_3a_4sa_3a_4] \subseteq (SAA \cup SAASAA \cup SSAASAA] \\ &\subseteq (SAA] \subseteq L_{wi}(A). \end{aligned}$$

It follows that $x \in L_{wi}(A)$. Hence, $S \subseteq L_{wi}(A)$. This shows that $S = L_{wi}(A)$. Next, we will show that A is a minimal subset of S with the property $S = L_{wi}(A)$. Suppose that $S = L_{wi}(B)$ for some $B \subset A$. Then there exists $a \in A$ such that $a \notin B$. Since $a \in A \subseteq S = L_{wi}(B) = (B \cup BB \cup SBB] = (B] \cup (BB \cup SBB]$, we have $a \in (B]$ or $a \in (BB \cup SBB]$. If $a \in (B]$, then there exists $y \in B$ such that $a \leq y$. Thus, $a \leq_{Lwi} y$, and $a \neq y$ where $a, y \in A$. This contradicts to Lemma 4.4. If $a \in (BB \cup SBB]$, we have $a \leq z$ for some $z \in BB \cup SBB$. Thus, $z \in BB$ or $z \in SBB$. If $z \in BB$, then $z = b_1b_2$ for some $b_1, b_2 \in B \subset A$. Since $a \notin B$, then $a \neq b_1$ and $a \neq b_2$. Since $a \leq z$ and $z = b_1b_2$, it follows that $a \leq b_1b_2$, and so $a \leq_{Lwi} b_1b_2$. This contradicts to (2). If $z \in SBB$, then $z = sb_3b_4$ for some $b_3, b_4 \in B \subset A$ and $s \in S$. Since $a \notin B$, we have $a \neq b_3$ and $a \neq b_4$. Since $a \leq z$ and $z = sb_3b_4$, it implies that $a \leq sb_3b_4$, and so $a \leq_{Lwi} sb_3b_4$. This contradicts to (3). Thus, there is no proper subset B of A such that $S = L_{wi}(B)$. This shows that A is a right weak-interior base of S. □

Finally, we find a condition in order of a right weak-interior base to be a subsemigroup.

Theorem 4.2. *Let A be a right weak-interior base of an ordered semigroup S. Then A is a subsemigroup of S if and only if A satisfies the following conditions: for any $a, b \in A$, $ab = a$ or $ab = b$.*

Proof: Assume that A is a subsemigroup of S. Let $a, b \in A$. Suppose that $ab \neq a$ and $ab \neq b$. Let $ab = c$. Then $c \neq a$ and $c \neq b$. Since $c = ab \in (ab \cup Sab]$, it follows by Lemma 4.2 that $c = a$ or $c = b$, which is a contradiction. The converse statement is obvious. □

Example 4.4. *Consider the ordered semigroup $S = \{a, b, c, d, e\}$ under the binary operation $\cdot$ and the partial order relation $\leq$ below:*

$\cdot$	a	b	c	d	e
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
e	e	e	e	e	e

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (b, e)\}.$$

Consider the subset $A = \{a, c\}$ of S. We obtain that $A = \{a, c\}$ is a right weak-interior base of S. Moreover, it is easy to see that A is a subsemigroup of S, and for $a, c \in A$, it follows that $ac = c$.

5. **Conclusions.** In this paper, we focus only on the results for left weak-interior ideals and right weak-interior bases of ordered semigroups. In Theorem 4.1, we give the necessary and sufficient condition for element in ordered semigroups, to be a right weak-interior base. In Theorem 4.2, we investigate the necessary and sufficient condition for right weak-interior base in ordered semigroups, to be a subsemigroup. Moreover, some results in [4] are special cases of the results of this paper. In the future work, we can study other results in this ordered algebraic structure. For example, we can extend this result to left weak-interior hyperideals and right weak-interior bases of ordered semihypergroups.

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