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CHARACTERIZATIONS OF ORDERED ALMOST IDEALS AND FUZZIFICATIONS IN PARTIALLY ORDERED TERNARY SEMIGROUPS

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Abstract. *The concept of ideal theory plays a vital role in algebraic structures. A partially ordered ternary semigroup is an algebraic structure under the ternary operator and a partial order satisfying compatible laws. This algebraic structure is generalizations of ordered semigroups and ternary semigroups. In this paper, we focus on studying the ideal theory of partially ordered ternary semigroups. We define ordered almost ideals and fuzzy ordered almost ideals in partially ordered ternary semigroups and we study some of their properties. We give relationships between fuzzy ordered almost ideals and ordered almost ideals of ordered semigroups.*
Keywords: Partially ordered ternary semigroups, Ordered almost ideals, Fuzzy ordered almost ideals

1. **Introduction.** Ternary algebraic systems were first studied by Kanser [1], who gave the idea of n-ary algebras. A ternary semigroup is a non-empty set together with an associative ternary operation. Every semigroup can be reduced to a ternary semigroup, but a ternary semigroup generally need not necessarily reduce to a semigroup. One of generalizations of a ternary semigroup, is so-called a partially ordered ternary semigroup (Shortly: po-ternary semigroup). A partially ordered ternary semigroup sometimes was said to be an ordered ternary semigroup. Moreover, the structure of partially ordered ternary semigroups generalizes the structure of ordered semigroups. Theory for ternary semigroups or ordered semigroups was extended to partially ordered ternary semigroups;

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we can see in [2, 3, 4, 5, 6, 7]. An ideal of a ring is a special subset of its elements. The study of ideals in rings is one of the important fields of research in ring theories. Similarly, an ideal theory in some algebraic structures is one of the important research fields. The notion of almost ideals (or A-ideals) of semigroups was introduced by Grosek and Satko [8] in 1980. The fuzzy sets were introduced by Zadeh [9]. The concept of fuzzy sets was widely studied in many fields. In 2018, Wattanatripop et al. [10] introduced the notion of fuzzy almost ideals in semigroups by using the concept of almost ideals and fuzzy ideals of semigroups. In [11], Suebsung et al. defined and studied some properties of almost ideals and fuzzy almost ideals of ternary semigroups. Recently, almost ordered ideals and their fuzzifications of ordered semigroups were studied in [12] and [13]. In this paper, we consider almost ordered ideals and their fuzzifications of partially ordered ternary semigroups. The results of this paper will extend some results of almost ideal theory in ternary semigroups and ordered semigroups. The main results are contained in Section 3. The main results are divided into 2 subsections.

(1) In Subsection 3.1, we focus on almost ordered ideals and their properties.
(2) In Subsection 3.2, we study fuzzifications of almost ordered ideals and investigate the relationships between almost ordered ideals and their fuzzifications.

Note that the results of this paper in [11] and [13] will be the special case of the results of this paper. Firstly, we will recall some definitions and results of partially ordered ternary semigroups in Section 2.

2. **Preliminaries.** In this section, we recall some definitions and results that are used throughout the paper.

A non-empty set T is called a *ternary semigroup* if there exists a ternary operation $[\,] : T \times T \times T \to T$, written as $(a, b, c) \longmapsto [abc]$, such that $[[abc]de] = [a[bcd]e] = [ab[cde]]$ for all $a, b, c, d, e \in T$. For non-empty subsets A, B and C of a ternary semigroup $(T, [\,])$, we let $[ABC] := \{[abc] \mid a \in A, b \in B, c \in C\}$. For simplicity, a ternary operation $[\,]$ will be identified with a multiplication of three elements, i.e., $[abc]$ will be identified with abc (in case a, b and c are elements) and $[ABC]$ will be identified with ABC (in case A, B and C are sets). In the case $A = \{a\}$, we write $\{a\}BC$ as aBC and similarly, in the cases $B = \{b\}$ or $C = \{c\}$, we respectively write AbC and ABc.

A *partially ordered ternary semigroup* $(T, [\,], \leq)$ is a ternary semigroup $(T, [\,])$ together with a partial order relation $\leq$ on T which is compatible with the ternary operation, i.e., $a \leq b \Rightarrow axy \leq bxy$, $xay \leq xby$ and $xya \leq xyb$ for all $a, b, x, y \in T$. Throughout this paper, we will write T for a partially ordered ternary semigroup, unless specified otherwise.

Let T be a partially ordered ternary semigroup. For a non-empty subset A of T, we denote $(A]$ to be the subset of T which is defined by $(A] := \{t \in T \mid t \leq a$ for some $a \in A\}$. If $A = \{a\}$, then we write $(\{a\}]$ as $(a]$. A non-empty subset A of T is called a *left (right, lateral) ordered ideal* of T if (1) $TTA \subseteq A$ ($ATT \subseteq A$, $TAT \subseteq A$) and (2) $A = (A]$, that is, for any $x \in A$ and $y \in T$, $y \leq x$ implies $y \in A$. If A is a left, right and lateral ordered ideal of T, then A is called an *ordered ideal* of T.

A function f from a set S to the unit interval $[0, 1]$ is called a *fuzzy subset* of S. The value of $f(x)$ is the value of the membership of x. For any two fuzzy subsets f and g of S, define the *union* and *intersection* of f and g, denoted by $f \cup g$ and $f \cap g$, by for all $x \in S$, $(f \cup g)(x) = \max\{f(x), g(x)\}$ and $(f \cap g)(x) = \min\{f(x), g(x)\}$. Let f and g be two fuzzy subsets of a partially ordered ternary semigroup T. Define $f \subseteq g$ by $f \subseteq g \iff f(x) \leq g(x)$ for all $x \in T$. For a fuzzy subset f of T, the *support* of f is defined by $supp(f) = \{x \in T | f(x) \neq 0\}$. The *characteristic mapping* of a subset A of T is a fuzzy subset of T defined by

$$C_A(x) = \begin{cases} 1; & x \in A, \\ 0; & x \notin A. \end{cases}$$

The definition of fuzzy points was given by Pu and Lui [14]. Let $t \in T$ and $\alpha \in (0,1]$. A *fuzzy point* t_α of T is a fuzzy subset of T defined by

$$t_\alpha(x) = \begin{cases} \alpha; & x = t, \\ 0; & x \neq t. \end{cases}$$

The value of α is the value of the membership of a point t. Next, let $F(T)$ be the set of all fuzzy subsets of T. For any $f, g, h \in F(T)$, we define the *product* of f, g and h by

$$(f \circ g \circ h)(x) := \begin{cases} \sup_{x \leq uvw} \min\{f(u), g(v), h(w)\} & \text{if } x \leq uvw \text{ where } u, v, w \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Let T be a partially ordered ternary semigroup. A fuzzy subset f of T is called a *fuzzy left (resp. right, lateral) ordered ideal* of T if for all $x, y, z \in T$, (1) if $x \leq y$, then $f(x) \geq f(y)$ and (2) $f(xyz) \geq f(z)$ (resp. $f(xyz) \geq f(x)$, $f(xyz) \geq f(y)$). A fuzzy subset f of S is called a *fuzzy ordered ideal* of T if it is a fuzzy left, right and lateral ordered ideal of T.

3. Main Results.

3.1. Ordered almost ideals.

In this subsection, we introduce the definition of ordered almost ideals in partially ordered ternary semigroups and we investigate some of their properties.

Definition 3.1. *Let T be a partially ordered ternary semigroup.*

(1) *A nonempty subset A of T is called a* left ordered almost ideal *of T if $(ttA] \cap A \neq \emptyset$ for all $t \in T$.*
(2) *A nonempty subset A of T is called a* right ordered almost ideal *of T if $(Att] \cap A \neq \emptyset$ for all $t \in T$.*
(3) *A nonempty subset A of T is called a* lateral ordered almost ideal *of T if $(tAt] \cap A \neq \emptyset$ for all $t \in T$.*
(4) *A nonempty subset A of T is called an* ordered almost ideal *of T if it is a left, right and lateral ordered almost ideal of T.*

Proposition 3.1. *Let T be a partially ordered ternary semigroup. The following statements are ture.*

(1) *Every left ordered ideal of T is a left ordered almost ideal of T.*
(2) *Every right ordered ideal of T is a right ordered almost ideal of T.*
(3) *Every lateral ordered ideal of T is a lateral ordered almost ideal of T.*
(4) *Every ordered ideal of T is an ordered almost ideal of T.*

Proof: We prove that (1) holds. Let A be a left ordered ideal of T and $t \in T$. Then $ttA \neq \emptyset$ and $ttA \subseteq A$. So $\emptyset \neq (ttA] \subseteq (A] \subseteq A$. Thus, $(ttA] \cap A \neq \emptyset$. Hence, A is a left ordered almost ideal of S. The proofs of (2) and (3) can be seen in the same manner. The statement (4) follows from (1), (2) and (3). □

The converse of Proposition 3.1(1), is not true in general as shown in the next example.

Example 3.1. *Consider the partially ordered ternary semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the partial order $\leq := \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1}), (\bar{2}, \bar{2}), (\bar{3}, \bar{3}), (\bar{4}, \bar{4}), (\bar{5}, \bar{5})\}$. Let $A = \{\bar{1}, \bar{3}, \bar{4}\}$. We have*

$(\bar{0} + \bar{0} + A] \cap A = A \cap A = A,$
$(\bar{1} + \bar{1} + A] \cap A = \{\bar{3}, \bar{5}, \bar{0}\} \cap A = \{\bar{3}\},$

$(\bar{2}+\bar{2}+A] \cap A = \{\bar{5},\bar{1},\bar{2}\} \cap A = \{\bar{1}\}$,
$(\bar{3}+\bar{3}+A] \cap A = A \cap A = A$,
$(\bar{4}+\bar{4}+A] \cap A = \{\bar{3},\bar{5},\bar{0}\} \cap A = \{\bar{3}\}$,
$(\bar{5}+\bar{5}+A] \cap A = \{\bar{5},\bar{1},\bar{2}\} \cap A = \{\bar{1}\}$.
Then $(t+t+A] \cap A \neq \emptyset$ for all $t \in \mathbb{Z}_6$. Thus, A is a left ordered almost ideal of $\mathbb{Z}_6$. However, A is not a left ordered ideal of $\mathbb{Z}_6$.

Theorem 3.1. *Let T be a partially ordered ternary semigroup. Then the following conditions hold.*

(1) *If A is a left ordered almost ideal of T, then every subset A' of T such that $A \subseteq A'$ is a left ordered almost ideal of T.*
(2) *If A is a right ordered almost ideal of T, then every subset A' of T such that $A \subseteq A'$ is a right ordered almost ideal of T.*
(3) *If A is a lateral ordered almost ideal of T, then every subset A' of T such that $A \subseteq A'$ is a lateral ordered almost ideal of T.*
(4) *If A is an ordered almost ideal of T, then every subset A' of T such that $A \subseteq A'$ is an ordered almost ideal of T.*

Proof: (1) Assume that A is a left ordered almost ideal of T. Let A' be a subset of T such that $A \subseteq A'$ and $t \in T$. Then $ttA \subseteq ttA'$, so $(ttA] \subseteq (ttA']$. This implies that $\emptyset \neq (ttA] \cap A \subseteq (ttA'] \cap A'$. Thus, A' is a left ordered almost ideal of T. The proofs of (2) and (3) are similar to that of (1). By statements (1), (2) and (3), it follows that the statement (4) holds. □

Corollary 3.1. *Let T be a partially ordered ternary semigroup. The following statements are true.*

(1) *If A_1 and A_2 are left ordered almost ideals of T, then $A_1 \cup A_2$ is a left ordered almost ideal of T.*
(2) *If A_1 and A_2 are right ordered almost ideals of T, then $A_1 \cup A_2$ is a right ordered almost ideal of T.*
(3) *If A_1 and A_2 are lateral ordered almost ideals of T, then $A_1 \cup A_2$ is a lateral ordered almost ideal of T.*
(4) *If A_1 and A_2 are ordered almost ideals of T, then $A_1 \cup A_2$ is an ordered almost ideal of T.*

Proof: Each of (1), (2), (3) and (4) follows from each of coresponding items in Theorem 3.1. □

For in case the intersection of two left ordered almost ideals is not true as shown in the following example.

Example 3.2. *Consider the ordered semigroup $\mathbb{Z}_6 = \{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the partial order $\leq := \{(\bar{0},\bar{0}),(\bar{1},\bar{1}),(\bar{2},\bar{2}),(\bar{3},\bar{3}),(\bar{4},\bar{4}),(\bar{5},\bar{5})\}$. Let $A_1 = \{\bar{1},\bar{3},\bar{4}\}$ and $A_2 = \{\bar{2},\bar{3},\bar{5}\}$. By Example 3.1, A_1 is a left ordered almost ideal of $\mathbb{Z}_6$. Similarly, it is easy to show that A_2 is a left ordered almost ideal of $\mathbb{Z}_6$, too. However, $A_1 \cap A_2 = \{\bar{3}\}$ is not a left ordered almost ideal of $\mathbb{Z}_6$ because $(\bar{1}+\bar{1}+(A_1 \cap A_2)] \cap (A_1 \cap A_2) = (\bar{1}+\bar{1}+\{\bar{3}\}] \cap \{\bar{3}\} = \{\bar{5}\} \cap \{\bar{3}\} = \emptyset$.*

Lemma 3.1. *Let T be a partially ordered ternary semigroup and $|T| > 1$. Then the following properties hold.*

(1) *T has no proper left ordered almost ideals if and only if for all $a \in T$, there exists an element $t_a \in T$ such that $(t_a t_a(T - \{a\})] = \{a\}$.*
(2) *T has no proper right ordered almost ideals if and only if for all $a \in T$, there exists an element $t_a \in T$ such that $((T - \{a\})t_a t_a] = \{a\}$.*

(3) *T has no proper lateral ordered almost ideals if and only if for all $a \in T$, there exists an element $t_a \in T$ such that $(t_a(T - \{a\})t_a] = \{a\}$.*

Proof: We prove that statement (1) holds. Assume T has no proper left ordered almost ideals and let $a \in T$. Then $T - \{a\}$ is not a left ordered almost ideal of T. That is there exists $t_a \in T$ such that $(t_a t_a(T - \{a\})] \cap (T - \{a\}) = \emptyset$. Thus, $(t_a t_a(T - \{a\})] = \{a\}$.

Conversely, let A be a proper nonempty subset of T. Then $A \subseteq T - \{a\}$ for some $a \in T$. By assumption, there exists $t_a \in T$ such that $(t_a t_a(T - \{a\})] = \{a\}$. Since $t_a t_a A \subseteq t_a t_a(T - \{a\})$, it follows that $(t_a t_a A] \subseteq (t_a t_a(T - \{a\})]$. We have that

$$(t_a t_a A] \cap A \subseteq (t_a t_a(T - \{a\})] \cap (T - \{a\}) = \{a\} \cap (T - \{a\}) = \emptyset.$$

Thus, $(t_a t_a A] \cap A = \emptyset$ which implies that A is not a left ordered almost ideal of T. Therefore, T has no proper left ordered almost ideals. Statements (2) and (3) can be proved in a similar manner. □

3.2. **Fuzzy ordered almost ideals.** In this subsection, we define fuzzy ordered almost ideals of partially ordered ternary semigroups and we examine some of their properties. Moreover, we provide the relationships between fuzzy ordered almost ideals and ordered almost ideals of partially ordered ternary semigroups.

Let T be a partially ordered ternary semigroup. For a fuzzy subset f of T, we defined $(f] : T \longrightarrow [0,1]$ by $(f](x) = \sup_{x \leq y} f(y)$ for all $x \in T$.

Proposition 3.2. *Let f, g and h be fuzzy subsets of a partially ordered ternary semigroup T. The following statements hold.*

(1) *$f \subseteq (f]$.*
(2) *If $f \subseteq g$, then $(f] \subseteq (g]$.*
(3) *If $f \subseteq g$, then $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$.*

Proof: (1) Let $x \in T$. We have $x \leq x$. Then $(f](x) = \sup_{x \leq y} f(y) \geq f(x)$. Thus, $f \subseteq (f]$.

(2) Assume that $f \subseteq g$. Then $f(x) \leq g(x)$ for all $x \in T$. Let $x \in T$. Thus, $(f](x) = \sup_{x \leq y} f(y) \leq \sup_{x \leq y} g(y) = (g](x)$. Hence, $(f] \subseteq (g]$.

(3) Assume that $f \subseteq g$. Thus, $f \circ h \subseteq g \circ h$ and $h \circ f \subseteq h \circ g$. It follows from (1) that, $(f \circ h] \subseteq (g \circ h]$ and $(h \circ f] \subseteq (h \circ g]$. □

Proposition 3.3. *Let f be a fuzzy subset of a partially ordered ternary semigroup T. The following statements are equivalent.*

(1) *If $x \leq y$, then $f(x) \geq f(y)$.*
(2) *$(f] = f$.*

Proof: Let $x \in T$. By assumption, $f(x) \geq f(y)$ for all $y \in T$ whenever $x \leq y$. Then $(f](x) = \sup_{x \leq y} f(y) = f(x)$. Hence, $(f] = f$. Conversely, we assume that $x, y \in T$ and $x \leq y$. Therefore, we have $f(x) = (f](x) = \sup_{x \leq y} f(y) \geq f(y)$. Thus, $f(x) \geq f(y)$. □

Definition 3.2. *Let T be a partially ordered ternary semigroup.*

(1) *A fuzzy subset f of T is called a* fuzzy left ordered almost ideal *of T if $(t_\alpha \circ t_\alpha \circ f] \cap f \neq 0$ for all $t \in T$ and $\alpha \in (0,1]$.*
(2) *A fuzzy subset f of T is called a* fuzzy right ordered almost ideal *of T if $(f \circ t_\alpha \circ t_\alpha] \cap f \neq 0$ for all $t \in T$ and $\alpha \in (0,1]$.*
(3) *A fuzzy subset f of T is called a* fuzzy lateral ordered almost ideal *of T if $(t_\alpha \circ f \circ t_\alpha] \cap f \neq 0$ for all $t \in T$ and $\alpha \in (0,1]$.*
(4) *A fuzzy subset f of T is called a* fuzzy ordered almost ideal *of T if f is a fuzzy left, right and lateral ordered almost ideal of T.*

Proposition 3.4. *Let f be a nonzero fuzzy subset of a partially ordered ternary semigroup T.*

(1) *Every fuzzy left ordered ideal of T is a fuzzy left ordered almost ideal of T.*
(2) *Every fuzzy right ordered ideal of T is a fuzzy right ordered almost ideal of T.*
(3) *Every fuzzy lateral ordered ideal of T is a fuzzy lateral ordered almost ideal of T.*
(4) *Every fuzzy ordered ideal of T is a fuzzy ordered almost ideal of T.*

Proof: (1) Assume that f is a fuzzy left ordered ideal of T. Let $t \in T$ and $\alpha \in (0,1]$. Since f is a nonzero function, there exists an element $a \in T$ such that $f(a) \neq 0$. Let $x = tta$. Since f is a fuzzy left ordered ideal, it follows that $f(x) = f(tta) \geq f(a) \neq 0$, so $f(x) \neq 0$. Then we see that

$$\begin{aligned}(t_\alpha \circ t_\alpha \circ f](x) &\geq \sup_{x \leq u}(t_\alpha \circ f)(u)\\ &\geq (t_\alpha \circ t_\alpha \circ f)(x)\\ &= \sup_{x \leq uvw} \min\{t_\alpha(u), t_\alpha(v), f(w)\}\\ &\geq \min\{t_\alpha(t), f(a)\} = \min\{\alpha, f(a)\} \neq 0.\end{aligned}$$

Thus, $((t_\alpha \circ t_\alpha \circ f] \cap f)(x) = \min\{(t_\alpha \circ t_\alpha \circ f](x), f(x)\} \neq 0$. Hence, f is a fuzzy left ordered almost ideal of T.

The proofs of (2) and (3), can be seen in the same manner.

(4) follows by (1), (2) and (3). □

Example 3.3. *Consider the partially ordered ternary semigroup $\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ under the addition on $\mathbb{Z}_6$ and the partial order $\leq := \{(a,a) | a \in \mathbb{Z}_6\}$. Define a function $f : \mathbb{Z}_6 \longrightarrow [0,1]$ by*

$$f(\bar{0}) = 0,\ f(\bar{1}) = 0.4,\ f(\bar{2}) = 0,\ f(\bar{3}) = 0,\ f(\bar{4}) = 0.3,\ f(\bar{5}) = 0.1.$$

Thus, for each $t \in \mathbb{Z}_6$ and $\alpha \in (0,1]$, there exists an element $x \in \mathbb{Z}_6$ such that $((t_\alpha \circ t_\alpha \circ f] \cap f)(x) \neq 0$. Hence, f is a fuzzy left ordered almost ideal of $\mathbb{Z}_6$ but not a fuzzy left ideal of $\mathbb{Z}_6$.

From the example above, it shows that the converse of Proposition 3.4(1) is not true in general.

Theorem 3.2. *Let T be a partially ordered ternary semigroup. The following statements are true.*

(1) *If f is a fuzzy left ordered almost ideal of T, then every fuzzy subset g of T such that $f \subseteq g$ is a fuzzy left ordered almost ideal of T.*
(2) *If f is a fuzzy right ordered almost ideal of T, then every fuzzy subset g of T such that $f \subseteq g$ is a fuzzy right ordered almost ideal of T.*
(3) *If f is a fuzzy lateral ordered almost ideal of T, then every fuzzy subset g of T such that $f \subseteq g$ is a fuzzy lateral ordered almost ideal of T.*
(4) *If f is a fuzzy ordered almost ideal of T, then every fuzzy subset g of T such that $f \subseteq g$ is a fuzzy ordered almost ideal of T.*

Proof: (1) Assume f is a fuzzy left ordered almost ideal of T. Let g be a fuzzy subset of T such that $f \subseteq g$ and let $t \in T$ and $\alpha \in (0,1]$. Then we have $t_\alpha \circ t_\alpha \circ f \subseteq t_\alpha \circ t_\alpha \circ g$ and so $(t_\alpha \circ t_\alpha \circ f] \subseteq (t_\alpha \circ t_\alpha \circ g]$. Thus, $(t_\alpha \circ t_\alpha \circ f] \cap f \subseteq (t_\alpha \circ t_\alpha \circ g] \cap g$. Since f is a fuzzy left ordered almost ideal, $(t_\alpha \circ t_\alpha \circ f] \cap f \neq \emptyset$. Hence, $(t_\alpha \circ t_\alpha \circ g] \cap g \neq \emptyset$. Therefore, g is a fuzzy left ordered almost ideal of T.

(2), (3) and (4) follow in the same way. □

Corollary 3.2. *Let T be a partially ordered ternary semigroup. The following statements are true.*

(1) *If f and g are fuzzy left ordered almost ideals of T, then $f \cup g$ is a fuzzy left ordered almost ideal of T.*

(2) *If f and g are fuzzy right ordered almost ideals of T, then $f \cup g$ is a fuzzy right ordered almost ideal of T.*
(3) *If f and g are fuzzy lateral ordered almost ideals of T, then $f \cup g$ is a fuzzy lateral ordered almost ideal of T.*
(4) *If f and g are fuzzy ordered almost ideals of T, then $f \cup g$ is a fuzzy ordered almost ideal of T.*

Proof: The proof follows from Theorem 3.2. □

Example 3.4. *Consider the partially ordered ternary semigroup $\mathbb{Z}_6 = \left\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\right\}$ under the addition on $\mathbb{Z}_6$ and the order $\leq := \{(a, a) | a \in \mathbb{Z}_6\}$. Define functions $f : \mathbb{Z}_6 \longrightarrow [0, 1]$ by*

$$f\left(\bar{0}\right) = 0,\ f\left(\bar{1}\right) = 0.6,\ f\left(\bar{2}\right) = 0,\ f\left(\bar{3}\right) = 0,\ f\left(\bar{4}\right) = 0.5,\ f\left(\bar{5}\right) = 0.2,$$

and $g : \mathbb{Z}_6 \longrightarrow [0, 1]$ by

$$g\left(\bar{0}\right) = 0,\ g\left(\bar{1}\right) = 0.7,\ g\left(\bar{2}\right) = 0.2,\ g\left(\bar{3}\right) = 0,\ g\left(\bar{4}\right) = 0,\ g\left(\bar{5}\right) = 0.4.$$

Thus, for each $t \in \mathbb{Z}_6$ and $\alpha \in (0, 1]$, there are elements $x, y \in \mathbb{Z}_6$ such that $((t_\alpha \circ t_\alpha \circ f] \cap f)(x) \neq 0$ and $((t_\alpha \circ t_\alpha \circ g] \cap g)(y) \neq 0$. Hence, f and g are fuzzy left ordered almost ideals of $\mathbb{Z}_6$.

Next consider $f \cap g : \mathbb{Z}_6 \longrightarrow [0, 1]$ where

$$(f \cap g)\left(\bar{0}\right) = 0,\ (f \cap g)\left(\bar{1}\right) = 0.6,\ (f \cap g)\left(\bar{2}\right) = 0,$$

$$(f \cap g)\left(\bar{3}\right) = 0,\ (f \cap g)\left(\bar{4}\right) = 0,\ (f \cap g)\left(\bar{5}\right) = 0.2.$$

It is easy to see that $f \cap g$ is not a fuzzy left ordered almost ideal of $\mathbb{Z}_6$.

Example 3.4 implies that the intersection of two fuzzy left ordered almost ideals need not be a fuzzy left ordered almost ideal.

Theorem 3.3. *Let A be a nonempty subset of a partially ordered ternary semigroup T. Then the following properties hold.*

(1) *A is a left ordered almost ideal of T if and only if C_A is a fuzzy left ordered almost ideal of T.*
(2) *A is a right ordered almost ideal of T if and only if C_A is a fuzzy right ordered almost ideal of T.*
(3) *A is a lateral ordered almost ideal of T if and only if C_A is a fuzzy lateral ordered almost ideal of T.*
(4) *A is an ordered almost ideal of T if and only if C_A is a fuzzy ordered almost ideal of T.*

Proof: (1) Assume A is a left ordered almost ideal of T. Let $t \in T$ and $\alpha \in (0, 1]$. Then $(ttA] \cap A \neq \emptyset$. That is there exists $x \in A$ and $x \in (ttA]$. So $C_A(x) = 1 \neq 0$ and $x \leq tta$ for some $a \in A$. We have

$$\begin{aligned}
(t_\alpha \circ t_\alpha \circ C_A](x) &= \sup_{x \leq y} (t_\alpha \circ t_\alpha \circ C_A)(y) \\
&\geq (t_\alpha \circ t_\alpha \circ C_A)(x) \\
&= \sup_{x \leq uvw} \min\{t_\alpha(u), t_\alpha(v), C_A(w)\} \\
&\geq \min\{t_\alpha(t), C_A(a)\} \\
&= \min\{\alpha, 1\} \neq 0.
\end{aligned}$$

Hence, $((t_\alpha \circ t_\alpha \circ C_A] \cap C_A)(x) \neq 0$. Therefore, C_A is a fuzzy left ordered almost ideal of T.

Conversely, assume that C_A is a fuzzy left ordered almost ideal of T. Let $t \in T$. There exists $x \in T$ such that $((t_\alpha \circ t_\alpha \circ C_A] \cap C_A)(x) \neq 0$ for all $\alpha \in (0, 1]$. That is $C_A(x) \neq 0$ and $(t_\alpha \circ t_\alpha \circ C_A](x) \neq 0$ for all $\alpha \in (0, 1]$. So $x \in A$. Since $(t_\alpha \circ t_\alpha \circ C_A](x) = \sup_{x \leq y}(t_\alpha \circ t_\alpha \circ C_A)(y) \neq 0$, there is $y \in T$ such that $x \leq y$ and

$$(t_\alpha \circ t_\alpha \circ C_A)(y) = \sup_{y \leq uvw} \min\{t_\alpha(u), t_\alpha(v), C_A(w)\} \neq 0.$$

Thus, $y \leq tta$ for some $a \in A$. Since $x \leq y$, we get $x \leq tta$, so $x \in (ttA]$. Hence, $x \in (ttA] \cap A$. Therefore, A is a left ordered almost ideal of T.

(2) and (3), can prove similarly to (1).

By properties (1), (2) and (3), it can see that property (4) holds. □

Theorem 3.4. *Let T be a partially ordered ternary semigroup. Then the following properties hold.*

(1) *f is a fuzzy left ordered almost ideal of T if and only if $supp(f)$ is a left ordered almost ideal of T.*
(2) *f is a fuzzy right ordered almost ideal of T if and only if $supp(f)$ is a right ordered almost ideal of T.*
(3) *f is a fuzzy lateral ordered almost ideal of T if and only if $supp(f)$ is a lateral ordered almost ideal of T.*
(4) *f is a fuzzy ordered almost ideal of T if and only if $supp(f)$ is an ordered almost ideal of T.*

Proof: (1) Assume that f is a fuzzy left ordered almost ideal of T. Let $t \in T$. There exists $x \in T$ such that $((t_\alpha \circ t_\alpha \circ f] \cap f)(x) \neq 0$ for all $\alpha \in (0, 1]$. That is $f(x) \neq 0$ and $(t_\alpha \circ t_\alpha \circ f](x) \neq 0$ for all $\alpha \in (0, 1]$. So $x \in supp(f)$. For all $\alpha \in (0, 1]$, there is an element $y \in T$ such that $x \leq y$ and $(t_\alpha \circ t_\alpha \circ f)(y) \neq 0$. Since $(t_\alpha \circ t_\alpha \circ f)(y) \neq 0$, we obtain $y \leq tta$ for some $a \in T$ such that $f(a) \neq 0$. Thus, $x \leq tta$ for some $a \in supp(f)$. This implies that $x \in (tt(supp(f))]$. Hence, $x \in (tt(supp(f))] \cap supp(f)$. Therefore, $supp(f)$ is a left ordered almost ideal of T.

Conversely, assume that $supp(f)$ is a left ordered almost ideal of T. By Theorem 3.3 (1), $C_{supp(f)}$ is a fuzzy left ordered almost ideal of T. Let $t \in T$ and $\alpha \in (0, 1]$. Then there exists $x \in T$ such that $\left(\left(t_\alpha \circ t_\alpha \circ C_{supp(f)}\right] \cap C_{supp(f)}\right)(x) \neq 0$ which implies that $C_{supp(f)}(x) \neq 0$ and $\left(t_\alpha \circ t_\alpha \circ C_{supp(f)}\right](x) \neq 0$. So $x \in supp(f)$ and there exists $y \in T$ such that $x \leq y$ and $\left(t_\alpha \circ t_\alpha \circ C_{supp(f)}\right)(y) \neq 0$. Thus, $y \leq tta$ for some $a \in supp(f)$. This implies that $x \leq tta$ where $f(x) \neq 0$ and $f(a) \neq 0$. Hence, $((t_\alpha \circ t_\alpha \circ f] \cap f)(x) \neq 0$. Therefore, f is a fuzzy left ordered almost ideal of T.

(2) and (3), can prove similarly to (1).

(4) follows from (1), (2) and (3). □

4. **Conclusions.**

In this paper, we define ordered almost ideals and fuzzy ordered almost ideals of partially ordered ternary semigroups. We show that the union of any two ordered almost ideals is also an ordered almost ideal but the intersection of any two ordered almost ideals need not be an ordered almost ideal. The case of fuzzy ordered almost ideals is similar. Moreover, we show the relationships between ordered almost ideals and their fuzzifications in Theorems 3.3 and 3.4.

In the future, we can extend some results of partially ordered semigroups and/or ternary semigroups to the results of partially ordered ternary semigroups.

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