ICIC Express Letters
Volume **17**, Number **6**, June **2023**
ICIC International ©2023 ISSN 1881-803X
pp. **685–694**

TENSORIZED HIGH-ORDER FUZZY SYSTEM FRAMEWORK AND ITS SUB-SYSTEMS WITH TENSOR UNFOLDING LEARNING ALGORITHM

Jie Li[1] and Guoliang Zhao[1,2,*]

[1]College of Electronic Information Engineering
[2]State Key Laboratory of Reproductive Regulation and Breeding of Grassland Livestock
Inner Mongolia University
No. 235, University West Road, Hohhot 010021, P. R. China
32056077@mail.imu.edu.cn; *Corresponding author: guoliangzhao@imu.edu.cn

Received October 2022; accepted January 2023

Abstract. *In this paper, tensorized high-order fuzzy system framework is studied. The interval type-n fuzzy system can be constructed under the proposed framework with tensor structure. Based on the proposed framework, three high-order fuzzy systems are proposed, which are tensorized type-2, type-3 and type-4 fuzzy systems. Antecedent part of tensorized type-n fuzzy system is a kind of interval type-n fuzzy set which is able to generate via the proposed framework. The whole antecedent part is combined with the $(n+2)$-dimensional tensor structure. The consequent part is completed by the tensor unfolding method and tensor's Moore-Penrose (M-P) inverse. The realization of the operation between tensors through Einstein product. Under the proposed framework, tensorized type-n fuzzy system has $(n+2)$-dimensional tensor structure, the consequent part unfolded $(n+2)$-dimensional tensor into 3^{n-2} $(n > 2)$ 4-dimensional tensors. Then, the consequent learning part is completed in high-order tensor space. Finally, the effectiveness of three kinds of models that are generated by the proposed framework is verified by simulation results. Simultaneously, the proposed framework is proved to be a feasible scheme for constructing high-order fuzzy sets and high-order fuzzy systems.*
Keywords: Tensorized fuzzy system, Interval type-n fuzzy set, High-order fuzzy system, Tensor regression

1. **Introduction.** Since Zadeh first proposed fuzzy set [1], fuzzy theory has developed for decades. Based on fuzzy set theory, various fuzzy systems have emerged and have been extended to many branches of science. Numerous studies have shown that approximation ability is considered to be the main ability of the fuzzy system. Therefore, fuzzy system is mainly used to estimate nonlinear functions or establish a mathematical model according to data set. Due to the excellent data modeling ability, fuzzy theory is employed in many scientific fields, such as expert systems [2], mobile robot control [3], binary classifier [4], fuzzy forecasting [5], sickness diagnosis [6], pattern recognition [7] and control system [8]. In the existing research on fuzzy system, type-1 fuzzy set (T1FS) and type-2 fuzzy set (T2FS) have been widely studied and applied. In [9], classification of faults is realized via a T1 fuzzy logic system in an electromechanical equipment. Recently, T1 and singleton fuzzy logic system are optimized via Broyden-Fletcher-Goldfarb-Shanno (BFGS) method in [4]. The performance of T1 fuzzy system in binary classification problems is improved. In [5], T1 fuzzy function is improved via Elastic-net regularization method, and this method is a new T1 fuzzy system optimization method. Compared with T1 fuzzy system, T2 fuzzy system is more widely studied. Mendel et al. introduced the T2 fuzzy logic systems [10]. In succession, IT2 fuzzy logic systems are proposed in [11] by team of Mendel. Afterwards, T2 fuzzy system has been extensively researched and applied. In

DOI: 10.24507/icicel.17.06.685

[12], the Gaussian interval type-2 fuzzy set (IT2FS) is applied for modeling the process of multi-attribute decision-making. In [13], T2FS is employed to enhance medical images' visibility. In [14], T2 fuzzy system is incorporated with things technology and used to appraise air quality. Type-reduction or defuzzification is the key content of type-2 fuzzy logic systems, and three kinds of center-of-sets type-reduction methods are compared in [15], recently.

In the aforementioned papers, it is shown that T1FS and T2FS have been fully studied and popularized in the past decades. However, there are a few relative researches on high-order fuzzy sets and high-order fuzzy systems. Zadeh introduced the content of high-order fuzzy set at the beginning of fuzzy set establishment [16]. Mendel et al. introduced and described the construction of type-n fuzzy set (TnFS) [17]. Subsequently, Rickard et al. proposed fuzzy subset in the light of type-n fuzzy set in [18]. A short introduction about the type-3 fuzzy set (T3FS) and high-order FSs is given in [19]. In [20], the interval type-3 (IT3) fuzzy system is introduced. After that, there are some research works around IT3 FS, such as MEMS gyroscopes' fuzzy predictive controller design [21], renewable energy modeling-prediction [22], nonlinear systems modeling [23], as well as 5G telecom applications and stabilization converter [24].

Although there are some researches on type-3 fuzzy system and type-n fuzzy set, there is no framework-like work on high-order fuzzy sets. In [25, 26, 27], the approach of constructing tensor-based T2 fuzzy system by combining tensor structure and T2FS is introduced. And performance and effectiveness of the algorithm are verified. Based on the tensor structure, a framework of the high-order fuzzy system is suggested in this work. The primary contributions of this work are as the following.

- In accordance with description of classical TnFS and high-order fuzzy theory, a high-order fuzzy system framework is studied in the tensor structure.
- Under the framework of tensorized high-order fuzzy system, tensorized type-2, type-3 and type-4 fuzzy systems are realized, and simulation experiments are carried out.
- The tensor unfolding learning algorithm is extended under the high-dimensional tensor structure, and the extended method is used in solving learning problem of the suggested framework.

The manuscript of this paper is as below. Background of type-n fuzzy set is expressed in Section 2. The design of the proposed tensorized high-order fuzzy system framework (THOFSF) is presented in Section 3, among which, Section 3.1 studies the tensorized high-order fuzzy set, and Section 3.2 introduces the consequent learning part of the suggested framework. Section 4 narrates the simulation test framework and discusses comparison results. At last, Section 5 gives conclusion part.

2. Background of High-Order Fuzzy Set.

2.1. Type-1 fuzzy set (T1FS).

T1FS is the classical content in fuzzy theory. In universe X, a T1FS can be denoted by Φ. And Φ is explained via its membership function (MF) $\mu_\Phi(x)$ which can be called type-1 MF. The elements of T1FS μ_Φ take values on interval $[0, 1]$. Therefore, T1FS is conveyed as a set of characteristics which can be extracted from interval $[0, 1]$ by membership function $\mu_\Phi(x)$. According to collection of points representation, the T1FS Φ is expressed as below [1, 16, 28, 29, 30]:

$$\Phi = \{(X, \mu_\Phi(x)) | x \in X\}, \tag{1}$$

where $\mu_\Phi : X$ for $[0, 1]$. In accordance with Formula (1), T1FS can be understood as the set of ordered pairs of a member.

2.2. **Type-2 fuzzy set (T2FS).** T2FS is constructed via T1FS. For the given general T2FS $\tilde{\Phi}$, it is denoted as below:

$$\tilde{\Phi} = \left\{ \left((x,u), \mu_{\tilde{\Phi}(x,u)} \right) | \forall x \in X, \forall u \in J_x \subseteq [0,1] \right\}, \tag{2}$$

where J_x is the primary membership of x. $0 \le \mu_{\tilde{\Phi}}(x,u) \le 1$ can be called the T2 MF and it can also be called secondary grades. The x is lied in range of primary MF. The u belongs to secondary MF. Additionally, the general T2FS $\tilde{\Phi}$ is defined as below [25]:

$$\tilde{\Phi} = \int_{x \in X} \int_{u \in J_x} \frac{\mu_{\tilde{\Phi}}(x,u)}{(x,u)}, \ J_x \subseteq [0,1], \tag{3}$$

where $\iint$ represents the association relation between x and u. Based on Formulae (2) and (3), secondary MF of general T2FS is a type-1 MF. And secondary grades comply this type-1 membership function. The IT2FS is deemed the exceptional condition of general T2FS. Similar to Formulae (2) and (3), the IT2FS can be defined as

$$\tilde{\Phi} = \{((x,u),1) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \text{ or } \tilde{\Phi} = \int_{x \in X} \int_{u \in J_x} \frac{1}{(x,u)}, \ J_x \subseteq [0,1], \tag{4}$$

where X is primary domain, and J_x is secondary domain. Obviously, the secondary membership grade $\mu_{\tilde{A}}(x,u)$ of IT2FS is equal to 1.

2.3. **Type-3 fuzzy set (T3FS).** Similar to the relationship between T1FS and T2FS, T3FS and T2FS inherit the similar relationship. In the light of the introduction of the high-order fuzzy set by Zadeh [16] and the notation of Mendel et al. [17], the type-n ($n > 1$) FSs can be described as the set of values extracted in type-($n-1$) FSs on $[0,1]$. That is, secondary MF of type-n FS is type-($n-1$) MF, and secondary grades of type-n FS are type-($n-1$) secondary grades.

Therefore, let $\tilde{\Phi}^3$ be an interval T3FS, which is expressed as [20]

$$\begin{aligned} \tilde{\Phi} &= \left\{ \left((x,u), \left[\frac{1}{\mu_{\tilde{\Phi}}(x,u)} \right] \right) \middle| \forall x \in X, \forall u \in J_x \subseteq [0,1] \right\} \\ \text{or } \ \tilde{\Phi} &= \int_{x \in X} \int_{u \in J_x} \frac{\left[\frac{1}{\mu_{\tilde{\Phi}}(x,u)} \right]}{(x,u)}, \ J_x \subseteq [0,1], \end{aligned} \tag{5}$$

where $\mu_{\tilde{\Phi}}(x,u)$ expresses interval type-2 MF.

3. The Proposed High-Order Fuzzy System Framework.

3.1. **Tensorized high-order fuzzy set.** In this section, tensorized high-order fuzzy set is proposed. The T1FS is the earliest and most classical fuzzy set. T2FS, T3FS and type-n FS are constructed by T1FS. In the content of tensor theory, fuzzy set can be combined in the tensor structure. In [25], triangular T2FS is combined in 4-dimensional tensor structure. In [26], the IT2FS extends the enhancement node of random vector functional link (RVFL) network, and the using IT2FS is denoted via tensor structure. On the basis of tensor model, a trapezoidal T2 fuzzy inference system (TT2FIS) is studied in [27], and trapezoidal T2FS is combined in tensor structure. Therefore, the effectiveness and implementation of reconstructing T2FS through tensor structure have been proved.

Tensorized T2 fuzzy set (T2FS) is the earliest tensor-based high-order fuzz set. Let $\mathcal{A}$ be a tensorized T2FS. The N-dimensional data set is expressed as $\{x\}_I = \{x_1, x_2, \ldots, x_I\}$, $x_I = [x_{i,1}, x_{i,2}, \ldots, x_{i,J}]^T \in \mathbb{R}^N$, where $i = 1, 2, \ldots, I$ and I is an order index of sorted set. The MFs of IT2FS are denoted as follows:

$$\bar{\mu}_{il} = \exp\left(-s_1 t^2\right), \quad \underline{\mu}_{il} = \exp\left(-s_2 t^2\right), \tag{6}$$

where $t = x_{ij} - m_{ij}$, $s_1 = \frac{1}{\bar{\sigma}_{ij}^2}$ and $s_2 = \frac{1}{\underline{\sigma}_{ij}^2}$ $(1 \le i \le I, 1 \le l \le L)$. m_{ij} is average value. $\bar{\sigma}_{ij}^2$ and $\underline{\sigma}_{ij}^2$ are standard deviation of UMF and LMF, respectively. The area between LMF and UMF is defined as foot of uncertainty (FOU) of the IT2FS. And the membership degree of IT2FS is 1.

For a given testing dataset $\{x_m, y_m\}_{t=1}^N$, the input sample $x_m = (x_{m_1}, x_{m_2}, \ldots, x_{mM}) \in \mathbb{R}^M$ and the desired output $y_m \in \mathbb{R}$. The lower MF of tensorized T2FS $\mathcal{A}$ is denoted $\underline{\mathcal{A}} \in \mathcal{R}^{N\times 2\times L\times 1}$. Meanwhile, it is noted that $\underline{\mathcal{A}}$ is an array of lower MFs of T2FS. And $\underline{\mathcal{A}}$ can be estimated the relationship between x_t and y_t. The array $\underline{\mathcal{A}}$ can be expressed as below:

$$\underline{\mathcal{A}}_{:,:,1,1} = \begin{bmatrix} \underline{\mu}(\varpi_{11}x_1 + \tau_{11}) & \underline{\mu}(\varpi_{12}x_1 + \tau_{12}) \\ \vdots & \vdots \\ \underline{\mu}(\varpi_{11}x_N + \tau_{11}) & \underline{\mu}(\varpi_{12}x_N + \tau_{12}) \end{bmatrix}, \ldots, \underline{\mathcal{A}}_{:,:,L,1} = \begin{bmatrix} \underline{\mu}(\varpi_{L1}x_1 + \tau_{L1}) & \underline{\mu}(\varpi_{L2}x_1 + \tau_{L2}) \\ \vdots & \vdots \\ \underline{\mu}(\varpi_{L1}x_N + \tau_{L1}) & \underline{\mu}(\varpi_{L2}x_N + \tau_{L2}) \end{bmatrix},$$

where $\varpi_{il} = [\varpi_{i1}, \varpi_{i2}, \ldots, \varpi_{iK}]$ are randomly generated input weights, $\mu(x_n) = [\mu(x_1), \mu(x_2), \ldots, \mu(x_N)]$ $(n = 1, 2, \ldots, N)$ are three similar membership function with different parameters (that is, $\underline{\mu}(\cdot)$, $\hat{\mu}(\cdot)$, and $\underline{\mu}(\cdot)$), and τ_{il} are stochastic generated bias. Similarly, the $\bar{\mathcal{A}} \in \mathcal{R}^{N\times 2\times L\times 1}$ can be denoted by upper membership function arrays, which are expressed as below:

$$\bar{\mathcal{A}}_{:,:,1,3} = \begin{bmatrix} \bar{\mu}(\varpi_{11}x_1 + \tau_{11}) & \bar{\mu}(\varpi_{12}x_1 + \tau_{12}) \\ \vdots & \vdots \\ \bar{\mu}(\varpi_{11}x_N + \tau_{11}) & \bar{\mu}(\varpi_{12}x_N + \tau_{12}) \end{bmatrix}, \ldots, \bar{\mathcal{A}}_{:,:,L,3} = \begin{bmatrix} \bar{\mu}(\varpi_{L1}x_1 + \tau_{L1}) & \bar{\mu}(\varpi_{L2}x_1 + \tau_{L2}) \\ \vdots & \vdots \\ \bar{\mu}(\varpi_{L1}x_N + \tau_{L1}) & \bar{\mu}(\varpi_{L2}x_N + \tau_{L2}) \end{bmatrix}.$$

The $\hat{\mathcal{A}} \in \mathcal{R}^{N\times 2\times L\times 1}$ can be denoted as principal membership function arrays, and it can be constructed in the same way as $\underline{\mathcal{A}}$ and $\bar{\mathcal{A}}$. $\hat{\mathcal{A}}$ can be expressed as below:

$$\hat{\mathcal{A}}_{:,:,1,2} = \begin{bmatrix} \hat{\mu}(\varpi_{11}x_1 + \tau_{11}) & \hat{\mu}(\varpi_{12}x_1 + \tau_{12}) \\ \vdots & \vdots \\ \hat{\mu}(\varpi_{11}x_N + \tau_{11}) & \hat{\mu}(\varpi_{12}x_N + \tau_{12}) \end{bmatrix}, \ldots, \hat{\mathcal{A}}_{:,:,L,2} = \begin{bmatrix} \hat{\mu}(\varpi_{L1}x_1 + \tau_{L1}) & \hat{\mu}(\varpi_{L2}x_1 + \tau_{L2}) \\ \vdots & \vdots \\ \hat{\mu}(\varpi_{L1}x_N + \tau_{L1}) & \hat{\mu}(\varpi_{L2}x_N + \tau_{L2}) \end{bmatrix}.$$

In conclusion, a 4-dimensional tensor $\mathcal{A} \in \mathbb{R}^{N\times 2\times L\times 3}$ is obtained via three 3-dimensional tensor which dimension is $N \times 2 \times L$. 4-dimensional tensor $\mathcal{A}$ can be constructed via T2FS. For convenience, $\mathcal{A}$ is called tensorized T2FS.

Remark 3.1. *The classical T2FS is constructed via lower MF, upper MF and principal MF. Principal MF is generated in type-reduction step of T2 fuzzy system. Because tensor structure integrates the content of the secondary MF directly, the reduction step of T2 fuzzy inference can be refrained in tensorized T2 fuzzy system modeling. Thus, tensorized fuzzy structure framework can directly combine the content of T2FS.*

In this paper, the proposed tensorized high-order system framework is designed as a multi-layer network. The tensorized T2FS described above can be regarded as the first layer of multi-layer network. The tensorized T3FS (T3FS) is the next layer, that is, the second layer of multi-layer network. As mentioned earlier, tensorized T2FS is constructed via T1FS. The tensorized T3FS of the proposed tensorized high-order system framework can be constructed via tensorized T2FS. For easy identification, the above tensorized T2FS $\mathcal{A}$ is denoted $\mathcal{A}^{(2)}$. Then, a tensorized T3FS can be expressed $\mathcal{A}^{(3)}$, and it can be defined as below:

$$\bar{\mathcal{A}}^{(3)}_{:,:,:,:,1} = \bar{\mathcal{A}}^{(2)},\ \hat{\mathcal{A}}^{(3)}_{:,:,:,:,2} = \hat{\mathcal{A}}^{(2)},\ \underline{\mathcal{A}}^{(3)}_{:,:,:,:,3} = \underline{\mathcal{A}}^{(2)}, \tag{7}$$

where $\bar{\mathcal{A}}^{(2)} \in \mathbb{R}^{N\times 2\times L\times 3}$, $\hat{\mathcal{A}}^{(2)} \in \mathbb{R}^{N\times 2\times L\times 3}$ and $\underline{\mathcal{A}}^{(2)} \in \mathbb{R}^{N\times 2\times L\times 3}$ represent upper T2 MF, secondary T2 MF and lower T2 MF, respectively. It is noted that the dimension of the suggested tensorized T3FS $\mathcal{A}^{(3)}$ is $N \times 2 \times L \times 3 \times 3$, that is, $\mathcal{A}^{(3)}$ is a 5-dimensional tensor. According to the introduction of T3FS in [18] and [20], when the secondary MF of T2FS changes from T1 to T2, the original T2FS becomes T3FS. That is, the secondary

MF of T3FS is T2 MF. The interval T3FS or tensorized T3FS $\mathcal{A}^{(2)}$ which is generated by Formula (7) has the same design with the interval T3FS in [18] and [20]. The secondary MF of tensorized T3FS is a T2 MF. The UMF and LMF of interval T3FS in [18] and [20] are T1 MF. However, the UMF and LMF of the proposed tensorized T3FS are both upgraded to T2 MF.

Similarly, the tensorized T4FS which is denoted as $\mathcal{A}^{(4)}$ can be constructed by tensorized T3FS. The tensorized T4FS $\mathcal{A}^{(4)}$ is expressed as follows:

$$\bar{\mathcal{A}}^{(4)}_{:,:,:,:,:,1} = \bar{\mathcal{A}}^{(3)},\ \hat{\mathcal{A}}^{(4)}_{:,:,:,:,:,2} = \hat{\mathcal{A}}^{(3)},\ \underline{\mathcal{A}}^{(4)}_{:,:,:,:,:,3} = \underline{\mathcal{A}}^{(3)}, \tag{8}$$

where $\mathcal{A}^{(4)} \in \mathbb{R}^{N\times 2\times L\times 3\times 3\times 3}$. $\bar{\mathcal{A}}^{(3)} \in \mathbb{R}^{N\times 2\times L\times 3\times 3}$, $\hat{\mathcal{A}}^{(3)} \in \mathbb{R}^{N\times 2\times L\times 3\times 3}$ and $\underline{\mathcal{A}}^{(3)} \in \mathbb{R}^{N\times 2\times L\times 3\times 3}$ are type-3 LMF, type-3 secondary MF and type-3 LMF, respectively. The dimension of $\mathcal{A}^{(4)}$ is $N \times 2 \times L \times 3 \times 3 \times 3$, that is, $\mathcal{A}^{(4)}$ is a 6-dimensional tensor.

In the above, the tensorized T2FS, tensorized T3FS and tensorized T4FS are described. For the proposed tensorized high-order system with multi-layer network structure, the first layer corresponds to tensorized T2FS, the second layer corresponds to tensorized T3FS, the third layer corresponds to tensorized T4FS and so on. In other words, the proposed tensorized type-n fuzzy system has $(n-1)$ layers network structure, and the tensorized type-n fuzzy set is constructed by tensorized type-$(n-1)$ set. Since then, the design of the antecedent part of the proposed tensorized high-order fuzzy system framework with multi-layer network structure has been completed. In the next section, the tensor regression for solving consequent learning part is introduced.

Remark 3.2. *According to the introduction of the high-order fuzzy set by Zadeh* [16] *and Mendel et al.* [17]*, type-n FS is ultimately composed by T1FS. For the proposed tensorized high-order fuzzy set, the promotion of order is on the basis of tensorized T2FS. And tensorized T2FS is based to T1FS. Thus, the proposed tensorized high-order fuzzy set conforms to the construction idea of classic high-order fuzzy theory.*

3.2. **Consequent learning part of the proposed framework.** In this section, the regression algorithm for solving consequent learning of the proposed framework is introduced. The mathematical model of the proposed high-order fuzzy system framework is

$$\mathcal{A}^{(n)} *_N \mathcal{X} = \mathcal{T}, \tag{9}$$

where $\mathcal{A}^{(n)} \in \mathbb{R}^{I_{1\cdots 2}\times J_{1\cdots 2}}$ is the tensor which is generated by tensorized type-n fuzzy system, where $I_{1\cdots 2} = I_1 \times I_2$, and $J_{1\cdots 2} = J_1 \times J_2$. $\mathcal{X} \in \mathbb{R}^{J_{1\cdots 2}}$ is the corresponding weight tensor and $\mathcal{T} \in \mathbb{R}^{I_{1\cdots 2}}$ is the output tensor. In [31], a generalized M-P inverse of tensor is proposed, which is utilized as regression algorithm of the proposed tensorized framework. The pseudo code of M-P inverse is shown in Algorithm 1.

Algorithm 1: Moore-Penrose inverse of tensor $\mathcal{A}$ [31]

Require: Input: Tensor $\mathcal{A}$, tensor index $P_{1\cdots K}$ and $Q_{1\cdots K}$, where $P_{1\cdots K} = P_1 \times \cdots \times P_K$, $Q_{1\cdots K} = Q_1 \times \cdots \times Q_K$.

Ensure: K is even, $P = \prod_{i=1}^{K} P_i$, $Q = \prod_{i=1}^{K} Q_i$.

1: Reshape tensor $\mathcal{A} \in \mathbb{R}^{P_{1\cdots K}\times Q_{1\cdots K}}$ into a matrix $A \in \mathbb{R}^{P\times Q}$.
2: Perform SVD on matrix A, A is decomposed as $A = FGH^*$, where H^* is conjugate transpose of H.
3: Reshaping matrix into tensor, that is, F into $\mathcal{F}$, G into $\mathcal{G}$, H^* into $\mathcal{H}^*$.
4: **Output:** Obtain Moore-Penrose inverse of $\mathcal{A}$ through $\mathcal{A}^+ = \mathcal{F} *_K \mathcal{G}^+ *_K \mathcal{H}^*$.

Then, the weight tensor $\mathcal{X}$ in Formula (9) is denoted as follows:

$$\mathcal{X} = \left\{\mathcal{A}^{(n)}\right\}^{+} *_N \mathcal{T}, \tag{10}$$

where $\left\{\mathcal{A}^{(n)}\right\}^{+}$ is M-P inverse of $\left\{\mathcal{A}^{(n)}\right\}$. With regard to tensorized type-2 fuzzy system, Formula (10) is rewritten as

$$\mathcal{X}^{(2)} = \left\{\mathcal{A}^{(2)}\right\}^{+} *_N \mathcal{T}, \tag{11}$$

where $\mathcal{A}^{(2)} \in R^{N\times 2\times L\times 3}$ is the regression tensor, and $\left\{\mathcal{A}^{(2)}\right\}^{+}$ is the M-P inverse of tensor $\mathcal{A}^{(2)}$. The output of the tensorized T2 fuzzy system in the suggested framework is indicated as below:

$$\hat{\mathcal{T}}^{(2)} = \mathcal{A}^{(2)} *_N \mathcal{X}^{(2)}. \tag{12}$$

For the tensorized type-3 fuzzy system, the regression tensor is $\mathcal{A}^{(3)} \in N\times 2\times L\times 3\times 3$. Tensor structure has the ability to expand with adjacent order tensors. In [27], tensor unfolding structure learning method is used to solve the consequent part, and a 4-D tensor is unfolded to generate three matrices. The effectiveness of tensor unfolding structure is verified. The regression tensor of tensorized type-3 fuzzy system in the proposed framework is a 5-D tensor. For consistency, the 5-D tensor $\mathcal{A}^{(3)}$ is expanded to generate three 4-D tensors in order to use the Moore-Penrose inverse for solving the learning part. Therefore, the output of tensorized type-3 fuzzy system in the proposed high-order fuzzy system framework can be expressed as below:

$$\hat{\mathcal{T}}^{(3)} = \frac{1}{3}\sum_{k=1}^{3} \mathcal{A}_k^{(2)} *_N \mathcal{X}_k^{(2)}, \tag{13}$$

where $\mathcal{A}_k^{(2)} \in \mathbb{R}^{N\times 2\times L\times 3}$, $k = 1, 2, 3$. Three 4-D tensors, $\mathcal{A}_1^{(2)}$, $\mathcal{A}_2^{(2)}$ and $\mathcal{A}_3^{(2)}$ can obtain from the 5-D tensor $\mathcal{A}^{(3)} \in \mathbb{R}^{N\times 2\times L\times 3\times 3}$. According to tensor unfolding method, $\mathcal{A}^{(3)}_{:,:,:,:,1}$ can be unfolded to $\mathcal{A}_1^{(2)}$, $\mathcal{A}^{(3)}_{:,:,:,:,2}$ can be unfolded to $\mathcal{A}_2^{(2)}$ and $\mathcal{A}^{(3)}_{:,:,:,:,3}$ can be unfolded to $\mathcal{A}_3^{(2)}$.

Finally, the output of tensorized type-n fuzzy system in the proposed high-order fuzzy system framework is denoted as below:

$$\hat{\mathcal{T}}^{(n)} = \frac{1}{3^{n-2}} \underbrace{\sum_{k_1=1}^{3} \cdots \sum_{k_{n-2}=1}^{3}}_{n-1\sum} \mathcal{A}^{(2)}_{k_1\cdots k_{n-2}} *_N \mathcal{X}^{(2)}_{k_1\cdots k_{n-2}}. \tag{14}$$

For example, the output of the tensorized type-4 fuzzy system in the proposed high-order fuzzy system framework can be expressed as below:

$$\hat{\mathcal{T}}^{(4)} = \frac{1}{9}\sum_{k_1=1}^{3}\sum_{k_2=1}^{3} \mathcal{A}^{(2)}_{k_1k_2} *_N \mathcal{X}^{(2)}_{k_1k_2}. \tag{15}$$

4. Simulation Experiment.

4.1. **Regression problems modeling.** Five regression problems datasets are conducted, which are Auto MPG, Concrete slump, Hybrid, Servo and Wine quality white. The parameters setting and attributes of each data set are shown in Table 1. The comparison outcomes of average value and standard deviation on each data set with 1000 times testing are tabulated in Table 2. In the proposed framework, tensorzied type-2 fuzzy system (THOFSF-T2), tensorzied type-3 fuzzy system (THOFSF-T3), and tensorzied type-4 fuzzy system (THOFSF-T4) are performed in simulation studies. Furthermore, random vector functional link (RVFL) network [32], trapezoidal T2 fuzzy inference system (TT-

Table 1. Basic information of five data sets

Data set	#Attributes	#Train sets	#Test sets
Auto MPG	7	274	118
Concrete slump	11	91	12
Hybrid	7	107	46
Servo	5	116	51
Wine quality white	12	3429	1469

Table 2. Comparison results with RVFL [32], TT2FIS [27], IT2FNN [33] and the proposed high-order fuzzy system framework

Data set	Phase (RMSE)	Statistic	Method					
			RVFL	IT2FNN	**TT2FIS**	THOFSF-T2	THOFSF-T3	THOFSF-T4
Auto MPG	Training	Mean	3.96e-01	4.35e-01	7.47e-01	3.83e-01	3.62e-01	**3.54e-01**
		Std	2.47e-02	2.51e-02	4.58e-02	2.29e-02	1.65e-02	**1.39e-02**
	Testing	Mean	4.50e-01	4.72e-01	7.37e-01	4.45e-01	4.15e-01	**4.04e-01**
		Std	4.51e-02	3.95e-02	**1.00e-15**	4.25e-02	3.59e-02	3.51e-02
Concrete slump	Training	Mean	9.23e-01	1.16e+00	1.43e+00	8.43e-01	7.29e-01	**6.89e-01**
		Std	1.16e-01	1.01e-01	1.07e-01	1.02e-01	6.76e-02	**5.18e-02**
	Testing	Mean	1.44e+00	1.39e+00	1.51e+00	1.46e+00	1.16e+00	**1.03e+00**
		Std	5.04e-01	2.08e-01	**5.33e-15**	3.81e-01	2.77e-01	1.73e-01
Hybrid	Training	Mean	9.23e-01	1.16e+00	1.43e+00	8.43e-01	7.29e-01	**6.89e-01**
		Std	1.16e-01	1.01e-01	1.07e-01	1.02e-01	6.76e-02	**5.18e-02**
	Testing	Mean	1.44e+00	1.39e+00	1.51e+00	1.46e+00	1.16e+00	**1.03e+00**
		Std	5.04e-01	2.08e-01	**5.33e-15**	3.81e-01	2.77e-01	1.73e-01
Servo	Training	Mean	5.96e-01	7.05e-01	1.27e+00	5.42e-01	4.93e-01	**4.78e-01**
		Std	7.71e-02	7.71e-02	7.13e-02	6.76e-02	5.70e-02	**5.45e-02**
	Testing	Mean	7.92e-01	8.38e-01	1.28e+00	7.81e-01	6.93e-01	**6.66e-01**
		Std	1.56e-01	1.63e-01	**7.11e-15**	1.60e-01	1.51e-01	1.56e-01
Wine quality white	Training	Mean	6.27e-01	6.39e-01	6.66e-01	6.30e-01	6.22e-01	**6.20e-01**
		Std	9.32e-03	1.02e-02	1.02e-02	9.74e-03	9.17e-03	**8.67e-03**
	Testing	Mean	6.54e-01	6.55e-01	6.47e-01	6.57e-01	6.47e-01	**6.44e-01**
		Std	2.19e-02	2.24e-02	**6.66e-16**	2.41e-02	2.22e-02	2.11e-02

2FIS) [27] and interval T2 fuzzy-neural network (IT2FNN) [33] are applied in the model comparisons.

As it can be seen, errors of RVFL, TT2FIS, IT2FNN and the proposed framework are listed in Table 2. THOFSF-T4 of the proposed high-order fuzzy system framework outperforms the other five models. For the RVFL and IT2FNN, the training and testing errors outperform TT2FIS on moderate scale data set Auto MPG, and three small scale datasets (Concrete slump, Hybird and Servo). And the performance of TT2FIS is better than RVFL and IT2FNN on large moderate scale data set Wine quality white. Thus, RVFL and IT2FNN are more suitable for deployment on moderate scale and small datasets scale, and TT2FIS is more suitable large moderate scale data set. Three kinds of sub-systems of the proposed high-order fuzzy system framework, which are THOFSF-T2, THOFSF-T3 and THOFSF-T4, obtained more excellent performance than RVFL, TT2FIS and IT2FNN on 5 different scales datasets. Performance of the suggested framework infers that it is feasible to construct high-order fuzzy system through tensor structure and tensor regression method. The effectiveness of the operation to construct the type-n fuzzy set is proved via the comparison results in Table 2. The results show that the proposed framework has better generalization performance and the ability to deal with high-order uncertain information.

4.2. **Nonlinear system fitting.** In this experiment, three kinds of sub-systems of the suggested high-order fuzzy system framework are used for nonlinear system fitting. The nonlinear system is given by [25], which can be denoted as below:

$$Y(s) = \frac{Y(s-1)Y(s-2)(Y(s-1)+2.5)}{1+(Y(s-1))^2+(Y(s-2))^2} + \zeta(s-1). \tag{16}$$

The system reaches equilibrium when it is in $(0,0)$. The input of fuzzy system is elected as $\zeta(s) \in \{-2,2\}$. Training input elects the uniformly distributed random variable in the $\{-2,2\}$. Testing input is $\zeta(s) = \sin(2\pi s/25)$. The nonlinear system (16) can be rewritten as below:

$$\hat{Y}(s) = \hat{f}(Y(s-1), Y(s-2), u(s-1)). \tag{17}$$

The fitting results are shown in Figure 1. And the results show that the proposed model has excellent performance. Excellent uncertain information modeling ability is an advantage of the fuzzy system. The THOFSF-T2, THOFSF-T3 and THOFSF-T4 are based on T2FS, T3FS and T4FS, respectively. The high-order fuzzy set has more complex structure, and it can include more uncertain information. Therefore, the uncertain information modeling ability of high-order fuzzy system is gradually enhanced with the increasing order of fuzzy system.

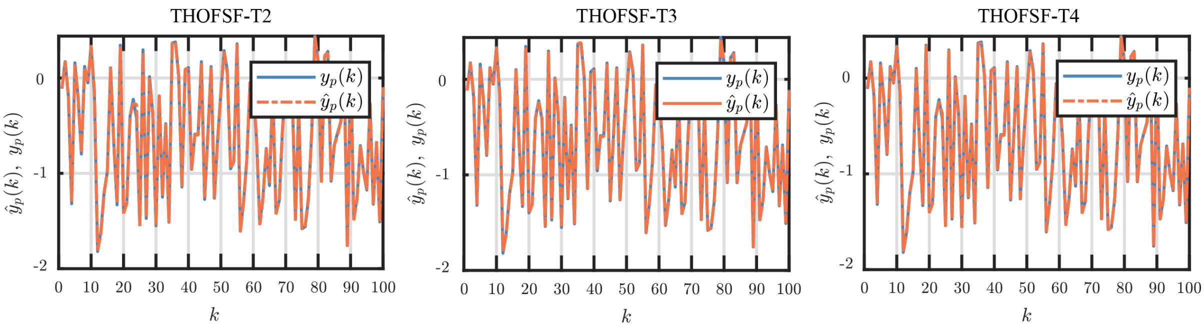

FIGURE 1. The prediction results of THOFSF-T2, THOFSF-T3 and THOFSF-T4 for the nonlinear system fitting problem

5. **Conclusion.** In this work, a tensorized high-order fuzzy system framework is studied. Based on tensor structure, the type-n fuzzy system can be constructed. In Section 3.1, tensorized T2FS, tensorized T3FS and tensorized T4FS are generated via the proposed framework. According to the tensor regression method and above three kinds of high-order fuzzy sets, the tensorized type-2, type-3 and type-4 fuzzy systems are constructed. Meanwhile, the formulaic output of the type-n fuzzy system which is generated via the proposed framework is given in Section 3.2. Two simulation experiments compare three fuzzy systems generated by the proposed framework with RVFL, IT2FNN and TT2FIS. RVFL can be considered as a T1 fuzzy network. IT2FNN can be considered as a T2 fuzzy neural network. TT2FIS is a T2 fuzzy system. The simulation results express that the suggested framework obtains excellent generalization ability and uncertainty modeling ability. And the proposed framework is a feasible scheme to construct type-n fuzzy sets and high-order fuzzy systems. There is still space for advancement in the suggested THOFSF in this study. In the future, the strategy of constructing tensorized high-order system will be updated and expanded around tensor structure and tensor regression algorithm.

Acknowledgement. This work is the results of the research project funded by the National Natural Science Foundation of China under number 62141305 and 12161065, Natural Science Foundation of Inner Mongolia (2020M-S06016, 2019MS01005).

REFERENCES

[1] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol.8, no.3, pp.338-353, 1965.

[2] L. A. Zadeh, The role of fuzzy logic in the management of uncertainty in expert systems, *Fuzzy Sets and Systems*, vol.11, no.1, pp.199-227, 1983.

[3] M. A. Sanchez, O. Castillo and J. R. Castro, Generalized type-2 fuzzy systems for controlling a mobile robot and a performance comparison with interval type-2 and type-1 fuzzy systems, *Expert Systems with Applications*, vol.42, no.14, pp.5904-5914, 2015.

[4] P. H. S. Calderano, C. R. Mateus Gheorghe, R. S. Teixeira, R. P. Finotti Amaral and I. F. M. Menezes, Type-1 and singleton fuzzy logic system binary classifier trained by BFGS optimization method, *Fuzzy Optimization and Decision Making*, 2022.

[5] N. Tak and D. İnan, Type-1 fuzzy forecasting functions with elastic net regularization, *Expert Systems with Applications*, vol.199, 116916, 2022.

[6] G. Manogaran, R. Varatharajan and M. K. Priyan, Hybrid recommendation system for heart disease diagnosis based on multiple kernel learning with adaptive neuro-fuzzy inference system, *Multimedia Tools and Applications*, vol.77, no.4, pp.4379-4399, 2018.

[7] T. Mahmood, U. Ur Rehman, Z. Ali and T. Mahmood, Hybrid vector similarity measures based on complex hesitant fuzzy sets and their applications to pattern recognition and medical diagnosis, *Journal of Intelligent & Fuzzy Systems*, vol.40, no.1, pp.625-646, 2021.

[8] B. Moaveni, F. R. Fathabadi and A. Molavi, Fuzzy control system design for wheel slip prevention and tracking of desired speed profile in electric trains, *Asian Journal of Control*, vol.24, no.1, pp.388-400, 2022.

[9] E. P. De Aguiar, M. A. Fernando, M. M. B. R. Vellasco and M. V. Ribeiro, Set-membership type-1 fuzzy logic system applied to fault classification in a switch machine, *IEEE Trans. Intelligent Transportation Systems*, vol.18, no.10, pp.2703-2712, 2017.

[10] N. N. Karnik, J. M. Mendel and Q. Liang, Type-2 fuzzy logic systems, *IEEE Trans. Fuzzy Systems*, vol.7, no.6, pp.643-658, 1999.

[11] Q. Liang and J. M. Mendel, Interval type-2 fuzzy logic systems: Theory and design, *IEEE Trans. Fuzzy Systems*, vol.8, no.5, pp.535-550, 2000.

[12] H. D. Wang, J. L. Yao, J. Yan and M. G. Dong, An extended topsis method based on Gaussian interval type-2 fuzzy set, *International Journal of Fuzzy Systems*, vol.21, no.6, pp.1831-1843, 2019.

[13] D. J. Bora and R. S. Thakur, An efficient technique for medical image enhancement based on interval type-2 fuzzy set logic, in *Progress in Computing, Analytics and Networking. Advances in Intelligent Systems and Computing*, P. Pattnaik, S. Rautaray, H. Das and J. Nayak (eds.), Singapore, Springer, 2018.

[14] A. Ghorbani and K. Zamanifar, Type-2 fuzzy ontology-based semantic knowledge for indoor air quality assessment, *Applied Soft Computing*, vol.121, 108658, 2022.

[15] C. Chen and J. X. Yang, Comparison studies of iterative algorithms for center-of-sets type-reduction of interval type-2 fuzzy logic systems, *International Journal of Innovative Computing, Information and Control*, vol.18, no.1, pp.29-39, 2022.

[16] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, *Information Sciences*, vol.8, no.3, pp.199-249, 1975.

[17] J. M. Mendel, H. Hagras and R. I. John, Standard background material about interval type-2 fuzzy logic systems that can be used by all authors, *IEEE Computational Intelligence Society Standard*, 2006.

[18] J. T. Rickard, J. Aisbett and G. Gibbon, Fuzzy subsethood for fuzzy sets of type-2 and generalized type-n, *IEEE Trans. Fuzzy Systems*, vol.17, no.1, pp.50-60, 2008.

[19] M. H. Fazel Zarandi, R. Gamasaee and O. Castillo, Type-1 to type-n fuzzy logic and systems, in *Fuzzy Logic in Its 50th Year: New Developments, Directions and Challenges*, C. Kahraman, U. Kaymak and A. Yazici (eds.), Cham, Springer, International Publishing, 2016.

[20] A. Mohammadzadeh, M. H. Sabzalian and W. Zhang, An interval type-3 fuzzy system and a new online fractional-order learning algorithm: Theory and practice, *IEEE Trans. Fuzzy Systems*, vol.28, no.9, pp.1940-1950, 2020.

[21] R. H. Vafaie, A. Mohammadzadeh and M. Piran, A new type-3 fuzzy predictive controller for MEMS gyroscopes, *Nonlinear Dynamics*, vol.106, no.1, pp.381-403, 2021.

[22] Y. Cao, A. Raise, A. Mohammadzadeh, S. Rathinasamy, S. S. Band and A. Mosavi, Deep learned recurrent type-3 fuzzy system: Application for renewable energy modeling/prediction, *Energy Reports*, vol.7, pp.8115-8127, 2021.

[23] A. Mohammadzadeh, O. Castillo, S. S. Band and A. Mosavi, A novel fractional-order multiple-model type-3 fuzzy control for nonlinear systems with unmodeled dynamics, *International Journal of Fuzzy Systems*, vol.23, no.6, pp.1633-1651, 2021.

[24] M. Gheisarnejad, A. Mohammadzadeh, H. Farsizadeh and M. H. Khooban, Stabilization of 5G telecom converter-based deep type-3 fuzzy machine learning control for telecom applications, *IEEE Trans. Circuits and Systems II: Express Briefs*, vol.69, no.2, pp.544-548, 2021.

[25] S. R. N. Huang, G. L. Zhao and M. H. Chen, Tensor extreme learning design via generalized Moore-Penrose inverse and triangular type-2 fuzzy sets, *Neural Computing and Applications*, vol.31, no.9, pp.5641-5651, 2019.

[26] G. L. Zhao and W. Wu, Tensor-based type-2 random vector functional link network, *2019 Chinese Control and Decision Conference (CCDC)*, pp.3278-3285, 2019.

[27] S. R. N. Huang, G. L. Zhao, Z. Weng and S. Y. Ma, Trapezoidal type-2 fuzzy inference system with tensor unfolding structure learning method, *Neurocomputing*, vol.473, pp.54-67, 2022.

[28] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-II, *Information Sciences*, vol.8, no.4, pp.301-357, 1975.

[29] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, *Information Sciences*, vol.9, no.1, pp.43-80, 1975.

[30] L. A. Zadeh, Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, vol.90, no.2, pp.111-127, 1997.

[31] R. Behera and D. Mishra, Further results on generalized inverses of tensors via the Einstein product, *Linear and Multilinear Algebra*, vol.65, no.8, pp.1662-1682, 2017.

[32] D. Husmeier, Random vector functional link (RVFL) networks, in *Neural Networks for Conditional Probability Estimation, Neural Networks for Conditional Probability Estimation*, London, Springer, 1999.

[33] C. F. Juang, R. B. Huang and W. Y. Cheng, An interval type-2 fuzzy-neural network with support-vector regression for noisy regression problems, *IEEE Trans. Fuzzy Systems*, vol.18, no.4, pp.686-699, 2010.