*ICIC Express Letters*
Volume **17**, Number **9**, September **2023**

ICIC International ©2023 ISSN 1881-803X
pp. 989–996

# ESSENTIAL INTERVAL VALUED FUZZY IDEALS IN SEMIGROUPS

Pannawit Khamrot[1] and Thiti Gaketem[2,*]

[1]Department of Mathematics
Faculty of Science and Agricultural Technology
Rajamangala University of Technology Lanna
Phitsanulok 65000, Thailand
pk_g@rmutl.ac.th

[2]Department of Mathematics
School of Science
University of Phayao
Phayao 56000, Thailand
*Corresponding author: thiti.ga@up.ac.th

Received November 2022; accepted January 2023

Abstract. *In this paper, we give the concepts of essential interval valued fuzzy ideals in semigroups. We proved properties and relationships between essential interval valued fuzzy ideals and essential ideals in semigroups. Moreover, we investigate properties of* 0*-essential interval valued fuzzy ideals and relationships between* 0*-essential interval valued fuzzy ideals and* 0*-essential ideals in semigroups.*

**Keywords:** Essential ideal, Essential interval valued fuzzy ideals, Interval valued fuzzy ideals, 0-essential ideal, 0-essential interval valued fuzzy ideals

1. **Introduction.** The concept of fuzzy sets was published by Zadeh in 1965 [1]. In 1979, Kuroki [2] studied a fuzzy semigroup and various kinds of fuzzy ideals in semigroups. In 1975, Zadeh [3] introduced the theory of interval valued fuzzy sets as a generalization of the notion of fuzzy sets. Interval valued fuzzy sets have various applications in several areas like medical science [4], image processing [5], and decision making [6]. Later, Narayanan and Manikantan studied the definition and properties of an interval valued fuzzy subsemigroup and an interval valued fuzzy ideal in semigroups in 2006 [7].

In 2008, Medhi et al. [8] studied essential fuzzy ideals of ring. In 2013, Medhi and Saikia [9] discussed the concept of T-fuzzy essential ideals and studied the properties of T-fuzzy essential ideals. In 2017, Wani and Pawar [10] extended the concept of essential ideals in semigroups to ternary semiring and studied essential ideals in ternary semiring. In 2021, Baupradist et al. [11] studied properties of the essential ideals and essential fuzzy ideals in semigroups. Together with 0-essential ideals and 0-essential fuzzy ideals in semigroups. In 2021, Chinram and Gaketem [12] give concept and investigated essential $(m, n)$-ideals and essential fuzzy $(m, n)$-ideals in semigroups. In 2023, Panpetch et al. [13] studied essential bi-ideals and fuzzy essential bi-ideals in semigroups. Moreover, Gaketem and Iampan [14,15] used knowledge of essential ideals in semigroups go to studied essential ideals in UP-algebra. Recently, Rittichuai et al. [16] studied properties of the essential ideals and essential fuzzy ideals in ternary semigroups.

In this papar, we used the concepts of essential fuzzy ideals to study interval valued fuzzy ideals in semigroups and investigated their properties. Moreover, we characterize essential interval valued fuzzy ideals and 0-essential interval valued fuzzy ideals of semigroups. Together, we prove relationship between essential interval fuzzy ideals and essential ideals in semigroups.

DOI: 10.24507/icicel.17.09.989

2. **Preliminaries.** In this section, we review concepts of basic definitions and the theorem used to prove all results in the next section.

A non-empty subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called a *subsemigroup* of $\mathcal{S}$ if $\mathcal{I}^2 \subseteq \mathcal{I}$. A non-empty subset $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called a *left (right)* ideal of $\mathcal{S}$ if $\mathcal{S}\mathcal{I} \subseteq \mathcal{I}$ ($\mathcal{I}\mathcal{S} \subseteq \mathcal{I}$). An *ideal* $\mathcal{I}$ of a semigroup $\mathcal{S}$ is a non-empty subset which is both a left ideal and a right ideal of $\mathcal{S}$. An ideal $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called an *essential ideal* of $\mathcal{S}$ if $\mathcal{I} \cap \mathcal{J} \neq \emptyset$ for every ideal $\mathcal{J}$ of $\mathcal{S}$.

We see that for any $\eta_1, \eta_2 \in [0,1]$, we have

$$\eta_1 \vee \eta_2 = \max\{\eta_1, \eta_2\} \text{ and } \eta_1 \wedge \eta_2 = \min\{\eta_1, \eta_2\}.$$

A fuzzy set $\eta$ of a non-empty set $\mathfrak{T}$ is function from $\mathfrak{T}$ into unit closed interval $[0,1]$ of real numbers, i.e., $\eta : \mathfrak{T} \to [0,1]$.

For any two fuzzy sets $\eta$ and $\nu$ of a non-empty set $\mathfrak{T}$, define $\geq, =, \wedge$ and $\vee$ as follows:

(1) $\eta \geq \nu \Leftrightarrow \eta(\mathfrak{k}) \geq \nu(\mathfrak{k})$ for all $\mathfrak{k} \in \mathfrak{T}$,
(2) $\eta = \nu \Leftrightarrow \eta \geq \nu$ and $\nu \geq \eta$,
(3) $(\eta \wedge \nu)(\mathfrak{k}) = \min\{\eta(\mathfrak{k}), \nu(\mathfrak{k})\} = \eta(\mathfrak{k}) \wedge \nu(\mathfrak{k})$ for all $\mathfrak{k} \in \mathfrak{T}$,
(4) $(\eta \vee \nu)(\mathfrak{k}) = \max\{\eta(\mathfrak{k}), \nu(\mathfrak{k})\} = \eta(\mathfrak{k}) \vee \nu(\mathfrak{k})$ for all $\mathfrak{k} \in \mathfrak{T}$.

For the symbol $\eta \leq \nu$, we mean $\nu \geq \eta$.

The following definitions are types of fuzzy subsemigroups on semigroups.

**Definition 2.1.** [17] *A fuzzy set $\eta$ of a semigroup $\mathcal{S}$ is said to be a fuzzy ideal of $\mathcal{S}$ if $\eta(\mathfrak{u}\mathfrak{v}) \geq \eta(\mathfrak{u}) \vee \eta(\mathfrak{v})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.*

**Definition 2.2.** [11] *An* essential fuzzy ideal *$\eta$ of a semigroup $\mathcal{S}$ if $\eta$ is a nonzero fuzzy ideal of $\mathcal{S}$ and $\eta \wedge \nu \neq 0$ for every nonzero fuzzy ideal $\nu$ of $\mathcal{S}$.*

Now, we review interval valued fuzzy set.

Let $C[0,1]$ be the set of all closed subintervals of $[0,1]$, i.e.,

$$C[0,1] = \left\{\breve{\mathfrak{p}} = \left[\mathfrak{p}^-, \mathfrak{p}^+\right] \middle| \, 0 \leq \mathfrak{p}^- \leq \mathfrak{p}^+ \leq 1\right\}.$$

We note that $[\mathfrak{p}, \mathfrak{p}] = \{\mathfrak{p}\}$ for all $\mathfrak{p} \in [0,1]$. For $\mathfrak{p} = 0$ or $1$ we shall denote $[0,0]$ by $\breve{0}$ and $[1,1]$ by $\breve{1}$.

Let $\breve{\mathfrak{p}} = [\mathfrak{p}^-, \mathfrak{p}^+]$ and $\breve{\mathfrak{q}} = [\mathfrak{q}^-, \mathfrak{q}^+] \in C[0,1]$. Define the operations $\preceq$, $=$, $\curlywedge$ and $\curlyvee$ as follows:

(1) $\breve{\mathfrak{p}} \preceq \breve{\mathfrak{q}}$ if and only if $\mathfrak{p}^- \leq \mathfrak{q}^-$ and $\mathfrak{p}^+ \leq \mathfrak{q}^+$,
(2) $\breve{\mathfrak{p}} = \breve{\mathfrak{q}}$ if and only if $\mathfrak{p}^- = \mathfrak{q}^-$ and $\mathfrak{p}^+ = \mathfrak{q}^+$,
(3) $\breve{\mathfrak{p}} \curlywedge \breve{\mathfrak{q}} = [(\mathfrak{p}^- \wedge \mathfrak{q}^-), (\mathfrak{p}^+ \wedge \mathfrak{q}^+)]$,
(4) $\breve{\mathfrak{p}} \curlyvee \breve{\mathfrak{q}} = [(\mathfrak{p}^- \vee \mathfrak{q}^-), (\mathfrak{p}^+ \vee \mathfrak{q}^+)]$.

If $\breve{\mathfrak{p}} \succeq \breve{\mathfrak{q}}$, we mean $\breve{\mathfrak{q}} \preceq \breve{\mathfrak{p}}$.

For each interval $\breve{\mathfrak{p}}_i = \left[\mathfrak{p}_i^-, \mathfrak{p}_i^+\right] \in C[0,1]$, $i \in \mathcal{A}$ where $\mathcal{A}$ is an index set, we define

$$\underset{i\in\mathcal{A}}{\curlywedge} \breve{\mathfrak{p}}_i = \left[\bigwedge_{i\in\mathcal{A}} \mathfrak{p}_i^-, \bigwedge_{i\in\mathcal{A}} \mathfrak{p}_i^+\right] \text{ and } \underset{i\in\mathcal{A}}{\curlyvee} \breve{\mathfrak{p}}_i = \left[\bigvee_{i\in\mathcal{A}} \mathfrak{p}_i^-, \bigvee_{i\in\mathcal{A}} \mathfrak{p}_i^+\right].$$

**Definition 2.3.** [7] *Let $\mathfrak{T}$ be a non-empty set. Then the function $\breve{\eta} : \mathfrak{T} \to C[0,1]$ is called* interval valued fuzzy set *(shortly, IVF set) of $\mathfrak{T}$.*

**Definition 2.4.** [7] *Let $\mathcal{L}$ be a subset of a non-empty set $\mathfrak{T}$. An* interval valued characteristic function *of $\mathfrak{T}$ is defined to be a function $\breve{\chi}_{\mathcal{L}} : \mathfrak{T} \to C[0,1]$ by*

$$\breve{\chi}_{\mathcal{L}}(\mathfrak{u}) = \begin{cases} \breve{1} & \text{if } \mathfrak{u} \in \mathcal{L}, \\ \breve{0} & \text{if } \mathfrak{u} \notin \mathcal{L} \end{cases}$$

*for all $e \in \mathfrak{T}$.*

For two IVF sets $\breve{\eta}$ and $\breve{\nu}$ of a non-empty set $\mathfrak{T}$, define

(1) $\breve{\eta} \sqsubseteq \breve{\nu} \Leftrightarrow \breve{\eta}(\mathfrak{u}) \preceq \breve{\nu}(\mathfrak{u})$ for all $\mathfrak{u} \in \mathfrak{T}$,
(2) $\breve{\eta} = \breve{\nu} \Leftrightarrow \breve{\eta} \sqsubseteq \breve{\nu}$ and $\breve{\nu} \sqsubseteq \breve{\eta}$,
(3) $(\breve{\eta} \sqcap \breve{\nu})(\mathfrak{u}) = \breve{\eta}(\mathfrak{u}) \curlywedge \breve{\nu}(\mathfrak{u})$ for all $\mathfrak{u} \in \mathfrak{T}$,
(4) $(\breve{\eta} \sqcup \breve{\nu})(\mathfrak{u}) = \breve{\eta}(\mathfrak{u}) \curlyvee \breve{\nu}(\mathfrak{u})$ for all $\mathfrak{u} \in \mathfrak{T}$.

For $\mathfrak{u} \in \mathcal{S}$, define $F_{\mathfrak{u}} = \{(\mathfrak{t}, \mathfrak{h}) \in \mathcal{S} \times \mathcal{S} \mid \mathfrak{u} = \mathfrak{t}\mathfrak{h}\}$.

For two IVF sets $\breve{\eta}$ and $\breve{\nu}$ in a semigroup $\mathcal{S}$, define the product $\breve{\eta} \circ \breve{\nu}$ as follows: for all $\mathfrak{u} \in \mathcal{S}$,

$$(\breve{\eta} \circ \breve{\nu})(\mathfrak{u}) = \begin{cases} \bigcup\limits_{\mathfrak{u}=\mathfrak{t}\mathfrak{h}} \{\breve{\eta}(\mathfrak{t}) \curlywedge \breve{\nu}(\mathfrak{h})\} & \text{if } F_{\mathfrak{u}} \neq \emptyset, \\ \breve{0} & \text{if } F_{\mathfrak{u}} = \emptyset. \end{cases}$$

**Definition 2.5.** [7] *An IVF subset $\breve{\eta}$ of a semigroup $\mathcal{S}$ is said to be an* IVF ideal *of $\mathcal{S}$ if $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \succeq \breve{\eta}(\mathfrak{u}) \curlyvee \breve{\eta}(\mathfrak{v})$ for all $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.*

The following theorems are true.

**Theorem 2.1.** [7] *Let $\mathcal{S}$ be a semigroup and let $\mathcal{L}$ be non-empty subset of $\mathcal{S}$. Then $\mathcal{L}$ is an ideal of $\mathcal{S}$ if and only if the characteristic function $\breve{\chi}_{\mathcal{L}}$ is an IVF ideal of $\mathcal{S}$.*

**Theorem 2.2.** [7] *Let $\mathcal{L}$ and $\mathcal{J}$ be subsets of a non-empty set $\mathcal{S}$. Then $\breve{\chi}_{\mathcal{L}\cap\mathcal{J}} = \breve{\chi}_{\mathcal{L}} \wedge \breve{\chi}_{\mathcal{J}}$ and $\breve{\chi}_{\mathcal{L}} \circ \breve{\chi}_{\mathcal{J}} = \breve{\chi}_{\mathcal{L}\mathcal{J}}$.*

The *support* of an IVF set $\breve{\eta}$ of a set is defined by $\operatorname{supp}(\breve{\eta}) = \{\mathfrak{u} \in \mathfrak{T} \,|\, \breve{\eta}(\mathfrak{u}) \neq \overline{0}\}$.

**Theorem 2.3.** *Let $\breve{\eta}$ be a nonzero IVF set of a semigroup $\mathcal{S}$. Then $\breve{\eta}$ is an IVF ideal of $\mathcal{S}$ if and only if* $\operatorname{supp}(\breve{\eta})$ *is an ideal of $\mathcal{S}$.*

**Proof:** Supposet that $\breve{\eta}$ is an IVF ideal of $\mathcal{S}$ and let $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$ with $\mathfrak{u}, \mathfrak{v} \in \operatorname{supp}(\breve{\eta})$. Then $\breve{\eta}(\mathfrak{u}) \neq \breve{0}$ and $\breve{\eta}(\mathfrak{v}) \neq \breve{0}$. Since $\breve{\eta}$ is an IVF ideal of $\mathcal{S}$, we have $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \succeq \breve{\eta}(\mathfrak{u}) \curlyvee \breve{\eta}(\mathfrak{v})$. Thus, $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \neq \breve{0}$, so $\mathfrak{u}\mathfrak{v} \in \operatorname{supp}(\breve{\eta})$. Hence, $\operatorname{supp}(\breve{\eta})$ is an ideal of $\mathcal{S}$.

Conversely, suppose that $\operatorname{supp}(\breve{\eta})$ is an ideal of $\mathcal{S}$ and let $\breve{\eta}$ be not an IVF ideal of $\mathcal{S}$. Then there exist $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$ such that $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \prec \breve{\eta}(\mathfrak{u}) \curlywedge \breve{\eta}(\mathfrak{v})$. Since $\operatorname{supp}(\breve{\eta})$ is an ideal of $\mathcal{S}$, we have $\mathfrak{u}\mathfrak{v} \in \operatorname{supp}(\breve{\eta})$. Thus, $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \neq \breve{0}$.

If $\mathfrak{u} \in \operatorname{supp}(\breve{\eta})$ or $\mathfrak{v} \in \operatorname{supp}(\breve{\eta})$, then $\breve{\eta}(\mathfrak{u}) \neq \breve{0}$ or $\breve{\eta}(\mathfrak{v}) \neq \breve{0}$ so $\mathfrak{u}\mathfrak{v} \in \operatorname{supp}(\breve{\eta})$. Thus, $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \succeq \breve{\eta}(\mathfrak{u}) \curlywedge \breve{\eta}(\mathfrak{v})$. It is a contradiction.

If $\mathfrak{u} \in \operatorname{supp}(\breve{\eta})$ and $\mathfrak{v} \in \operatorname{supp}(\breve{\eta})$, then $\breve{\eta}(\mathfrak{u}) \neq \breve{0}$ and $\breve{\eta}(\mathfrak{v}) \neq \breve{0}$ so $\mathfrak{u}\mathfrak{v} \in \operatorname{supp}(\breve{\eta})$. Thus, $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \succeq \breve{\eta}(\mathfrak{u}) \curlywedge \breve{\eta}(\mathfrak{v})$. It is a contradiction.

Hence, $\breve{\eta}(\mathfrak{u}\mathfrak{v}) \succeq \breve{\eta}(\mathfrak{u}) \curlyvee \breve{\eta}(\mathfrak{v})$. Therefore, $\breve{\eta}$ is an IVF ideal of $\mathcal{S}$. □

## 3. Essential Interval Valued Fuzzy Ideals in a Semigroup.

**Definition 3.1.** *A nonzero IVF ideal $\breve{\eta}$ of a semigroup $\mathcal{S}$ is called an* essential IVF ideal *of $\mathcal{S}$ if $\breve{\eta} \curlywedge \breve{\nu} \neq \breve{0}$ for every nonzero IVF ideal $\breve{\nu}$ of $\mathcal{S}$.*

**Theorem 3.1.** *Let $\mathcal{I}$ be an ideal of a semigroup $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$ if and only if $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal of $\mathcal{S}$.*

**Proof:** Suppose that $\mathcal{I}$ is an essential ideal of $\mathcal{S}$ and let $\breve{\nu}$ be a nonzero IVF ideal of $\mathcal{S}$. Then by Theorem 2.3, $\operatorname{supp}(\breve{\nu})$ is an ideal of $\mathcal{S}$. Since $\mathcal{I}$ is an essential ideal of $\mathcal{S}$, we have $\mathcal{I}$ is an ideal of $\mathcal{S}$. Thus, $\mathcal{I} \cap \operatorname{supp}(\breve{\nu}) \neq \emptyset$. So there exists $\mathfrak{u} \in \mathcal{I} \cap \operatorname{supp}(\breve{\nu})$. Since $\mathcal{I}$ is an ideal of $\mathcal{S}$, we have $\breve{\chi}_{\mathcal{I}}$ is an IVF ideal of $\mathcal{S}$. Since $\breve{\nu}$ is a nonzero IVF ideal of $\mathcal{S}$, we have $(\breve{\chi}_{\mathcal{I}} \curlywedge \breve{\nu})(\mathfrak{u}) \neq \breve{0}$. Thus, $\breve{\chi}_{\mathcal{I}} \curlywedge \breve{\nu} \neq \breve{0}$. Therefore, $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal of $\mathcal{S}$.

Conversely, assume that $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be an ideal of $\mathcal{S}$. Then $\breve{\chi}_{\mathcal{J}}$ is a nonzero IVF ideal of $\mathcal{S}$. Since $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal of $\mathcal{S}$, we have $\breve{\chi}_{\mathcal{I}}$ is an IVF ideal of $\mathcal{S}$. Thus, $\breve{\chi}_{\mathcal{I}} \curlywedge \breve{\chi}_{\mathcal{J}} \neq \breve{0}$. So by Theorem 2.2, $\breve{\chi}_{\mathcal{I}\cap\mathcal{J}} \neq \breve{0}$. Hence, $\mathcal{I} \cap \mathcal{J} \neq \emptyset$. Therefore, $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. □

**Theorem 3.2.** *Let $\breve{\eta}$ be a nonzero IVF ideal of a semigroup $\mathcal{S}$. Then $\breve{\eta}$ is an essential IVF ideal of $\mathcal{S}$ if and only if* $\operatorname{supp}(\breve{\eta})$ *is an essential ideal of $\mathcal{S}$.*

**Proof:** Assume that $\breve{\eta}$ is an essential IVF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be an ideal of $\mathcal{S}$. Then by Theorem 2.1, $\breve{\chi_{\mathcal{J}}}$ is an IVF ideal of $\mathcal{S}$. Since $\breve{\eta}$ is an essential IVF ideal of $\mathcal{S}$, we have $\breve{\eta}$ is an IVF ideal of $\mathcal{S}$. Thus, $\breve{\eta} \curlywedge \breve{\chi_{\mathcal{J}}} \neq \breve{0}$. So there exists $\mathfrak{u} \in \mathcal{S}$ such that $(\breve{\eta} \curlywedge \breve{\chi_{\mathcal{J}}})(\mathfrak{u}) \neq \breve{0}$. It implies that $\breve{\eta}(\mathfrak{u}) \neq 0$ and $\breve{\chi_{\mathcal{J}}}(\mathfrak{u}) \neq 0$. Hence, $\mathfrak{u} \in \operatorname{supp}(\breve{\eta}) \cap \mathcal{J}$, so $\operatorname{supp}(\breve{\eta}) \cap \mathcal{J} \neq \emptyset$. Therefore, $\operatorname{supp}(\breve{\eta})$ is an essential ideal of $\mathcal{S}$.

Conversely, assume that $\operatorname{supp}(\breve{\eta})$ is an essential ideal of $\mathcal{S}$ and let $\breve{\nu}$ be a nonzero IVF ideal of $\mathcal{S}$. Then by Theorem 2.3, $\operatorname{supp}(\breve{\nu})$ is an ideal of $\mathcal{S}$. Since $\operatorname{supp}(\breve{\eta})$ is an essential ideal of $\mathcal{S}$, we have $\operatorname{supp}(\breve{\eta})$ is an ideal of $\mathcal{S}$. Thus, $\operatorname{supp}(\breve{\eta}) \cap \operatorname{supp}(\breve{\nu}) \neq \emptyset$. So there exists $\mathfrak{u} \in \operatorname{supp}(\breve{\eta}) \cap \operatorname{supp}(\breve{\nu})$. It implies that $\breve{\eta}(\mathfrak{u}) \neq \breve{0}$ and $\breve{\nu}(\mathfrak{u}) \neq \breve{0}$ for all $\mathfrak{u} \in \mathcal{S}$. Hence, $(\breve{\eta} \curlywedge \breve{\nu})(\mathfrak{u}) \neq \breve{0}$ for all $\mathfrak{u} \in \mathcal{S}$. Therefore, $\breve{\eta} \curlywedge \breve{\nu} \neq \breve{0}$. We conclude that $\breve{\eta}$ is an essential IVF ideal of $\mathcal{S}$. □

**Theorem 3.3.** *Let $\breve{\eta}$ be an essential IVF ideal of a semigroup $\mathcal{S}$. If $\breve{\nu}$ is an IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\nu}$, then $\breve{\nu}$ is also an essential IVF ideal of $\mathcal{S}$.*

**Proof:** Let $\breve{\nu}$ be an IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\nu}$ and let $\breve{\mu}$ be any IVF ideal of $\mathcal{S}$. Since $\breve{\eta}$ is an essential IVF ideal of $\mathcal{S}$, we have $\breve{\eta}$ is an IVF ideal of $\mathcal{S}$. Thus, $\breve{\eta} \curlywedge \breve{\mu} \neq \breve{0}$. So $\breve{\nu} \curlywedge \breve{\mu} \neq \breve{0}$. Hence, $\breve{\nu}$ is an essential IVF ideal of $\mathcal{S}$. □

**Theorem 3.4.** *Let $\breve{\eta}_1$ and $\breve{\eta}_2$ be essential IVF ideals of a semigroup $\mathcal{S}$. Then $\breve{\eta}_1 \sqcup \breve{\eta}_2$ and $\breve{\eta}_1 \sqcap \breve{\eta}_2$ are essential IVF ideals of $\mathcal{S}$.*

**Proof:** Since $\breve{\eta}_2 \sqsubseteq \breve{\eta}_1 \sqcup \breve{\eta}_2$, we have $\breve{\eta}_1 \sqcup \breve{\nu}_2$ is an essential IVF ideal of $\mathcal{S}$.

Since $\breve{\eta}_1$ and $\breve{\eta}_2$, are essential IVF ideals of $\mathcal{S}$, we have $\breve{\eta}_1$ and $\breve{\eta}_2$ are IVF ideals of $\mathcal{S}$. Thus, $\breve{\eta}_1 \sqcap \breve{\eta}_2$ is an IVF ideal of $\mathcal{S}$. Let $\breve{\mu}$ be an IVF ideal of $\mathcal{S}$. Then $\breve{\eta}_1 \curlywedge \breve{\mu} \neq \breve{0}$. Thus, there exists $\mathfrak{u} \in \mathcal{S}$ such that $(\breve{\eta}_1 \curlywedge \breve{\mu})(\mathfrak{u}) \neq \breve{0}$. So $\breve{\eta}_1(\mathfrak{u}) \neq \breve{0}$ and $\breve{\mu}(\mathfrak{u}) \neq \breve{0}$. Since $\breve{\eta}_2 \neq \breve{0}$, let $\mathfrak{v} \in \mathcal{S}$ such that $\breve{\eta}_2(\mathfrak{v}) \neq \breve{0}$. Since $\breve{\eta}_1$ and $\breve{\eta}_2$ are IVF ideals of $\mathcal{S}$, we have $\breve{\eta}_1(\mathfrak{uv}) \succeq \breve{\eta}_1(\mathfrak{u}) \curlyvee \breve{\eta}_1(\mathfrak{v}) \geq \breve{0}$ and $\breve{\eta}_2(\mathfrak{uv}) \succeq \breve{\eta}_2(\mathfrak{u}) \curlyvee \breve{\eta}_2(\mathfrak{v}) \geq \breve{0}$. Thus, $(\breve{\eta}_1 \curlywedge \breve{\eta}_2)(\mathfrak{uv}) = \breve{\eta}_1(\mathfrak{uv}) \curlywedge \breve{\eta}_2(\mathfrak{uv}) \neq \breve{0}$. Since $\breve{\mu}$ is an IVF ideal of $\mathcal{S}$ and $\breve{\mu}(\mathfrak{u}) \neq \breve{0}$, we have $\breve{\mu}(\mathfrak{uv}) \neq \breve{0}$ for all $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$. Thus, $((\breve{\eta}_1 \curlywedge \breve{\eta}_2) \curlywedge \breve{\mu})(\mathfrak{uv}) \neq \breve{0}$. Hence, $((\breve{\eta}_1 \curlywedge \breve{\eta}_2) \curlywedge \breve{\mu}) \neq \breve{0}$. Therefore, $\breve{\eta}_1 \sqcap \breve{\eta}_2$ is an essential IVF ideal of $\mathcal{S}$. □

**Definition 3.2.** [11] *An essential ideal $\mathcal{I}$ of a semigroup $\mathcal{S}$ is called*
(1) *a* minimal *if for every essential ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$, we have $\mathcal{J} = \mathcal{I}$,*
(2) *a* prime *if $\mathfrak{uv} \in \mathcal{I}$ implies $\mathfrak{u} \in \mathcal{I}$ or $\mathfrak{v} \in \mathcal{I}$,*
(3) *a* semiprime *if $\mathfrak{u}^2 \in \mathcal{I}$ implies $\mathfrak{u} \in \mathcal{I}$, for all $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.*

**Example 3.1.** [11] *Let $\mathcal{S}$ be a semigroup with zero. Then $\{0\}$ is a unique minimal essential ideal of $\mathcal{S}$, since $\{0\}$ is an eseential ideal of $\mathcal{S}$.*

**Definition 3.3.** *An essential IVF ideal $\breve{\eta}$ of a semigroup $\mathcal{S}$ is called*
(1) *a* minimal *if for every essential IVF ideal of $\breve{\nu}$ of $\mathcal{S}$ such that $\breve{\nu} \sqsubseteq \breve{\eta}$, we have* $\operatorname{supp}(\breve{\eta}) = \operatorname{supp}(\breve{\nu})$*,*
(2) *a* prime *if $\breve{\eta}(\mathfrak{uv}) \preceq \breve{\eta}(\mathfrak{u}) \curlyvee \breve{\eta}(\mathfrak{v})$,*
(3) *a* semiprime *if $\breve{\eta}(\mathfrak{u}^2) \preceq \breve{\eta}(\mathfrak{u})$, for all $\mathfrak{u} \in \mathcal{S}$.*

**Theorem 3.5.** *Let $\mathcal{I}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds.*
(1) *$\mathcal{I}$ is a minimal essential ideal of $\mathcal{S}$ if and only if $\breve{\chi_{\mathcal{I}}}$ is a minimal essential IVF ideal of $\mathcal{S}$,*
(2) *$\mathcal{I}$ is a prime essential ideal of $\mathcal{S}$ if and only if $\breve{\chi_{\mathcal{I}}}$ is a prime essential IVF ideal of $\mathcal{S}$,*
(3) *$\mathcal{I}$ is a semiprime essential ideal of $S$ if and only if $\breve{\chi_{\mathcal{I}}}$ is a semiprime essential IVF ideal of $\mathcal{S}$.*

**Proof:**

(1) Suppose that $\mathcal{I}$ is a minimal essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. By Theorem 3.1, $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal of $\mathcal{S}$. Let $\breve{\eta}$ be an essential IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\chi}_{\mathcal{I}}$. Then $\text{supp}(\breve{\eta}) \sqsubseteq \text{supp}(\breve{\chi}_{\mathcal{I}})$. Thus, $\text{supp}(\breve{\eta}) \sqsubseteq \text{supp}(\breve{\chi}_{\mathcal{I}}) = \mathcal{I}$. Hence, $\text{supp}(\breve{\eta}) \sqsubseteq \mathcal{I}$. Since $\breve{\eta}$ is an essential IVF ideal of $\mathcal{S}$, we have $\text{supp}(\breve{\eta})$ is an essential ideal of $\mathcal{S}$. By assumption, $\text{supp}(\breve{\eta}) = \mathcal{I} = \text{supp}(\breve{\chi}_{\mathcal{I}})$. Hence, $\breve{\chi}_{\mathcal{I}}$ is a minimal essential IVF ideal of $\mathcal{S}$.

Conversely, $\breve{\chi}_{\mathcal{I}}$ is a minimal essential IVF ideal of $\mathcal{S}$ and let $\mathcal{B}$ be an essential ideal of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{I}$. Then $\mathcal{B}$ is an ideal of $\mathcal{S}$. Thus, by Theorem 3.1, $\breve{\chi}_{\mathcal{B}}$ is an essential IVF ideal of $\mathcal{S}$ such that $\breve{\chi}_{\mathcal{B}} \sqsubseteq \breve{\chi}_{\mathcal{I}}$. So $\breve{\chi}_{\mathcal{B}} = \breve{\chi}_{\mathcal{I}}$. Hence, $\mathcal{B} = \text{supp}(\breve{\chi}_{\mathcal{B}}) = \text{supp}(\breve{\chi}_{\mathcal{I}}) = \mathcal{I}$. Therefore, $\mathcal{I}$ is a minimal essential ideal of $\mathcal{S}$.

(2) Suppose that $\mathcal{I}$ is a prime essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Thus, by Theorem 3.1 $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal of $\mathcal{S}$. Let $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.

If $\mathfrak{u}\mathfrak{v} \in \mathcal{I}$, then $\mathfrak{u} \in \mathcal{I}$ or $\mathfrak{v} \in \mathcal{I}$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v}) = \breve{1} \succeq \breve{\chi}_{\mathcal{I}}(\mathfrak{u}\mathfrak{v})$.

If $\mathfrak{u}\mathfrak{v} \notin \mathcal{I}$, then $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}\mathfrak{v}) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v})$.

Thus, $\breve{\chi}_{\mathcal{I}}$ is a prime essential IVF ideal of $\mathcal{S}$.

Conversely, suppose that $\breve{\chi}_{\mathcal{I}}$ is a prime essential IVF ideal of $\mathcal{S}$. Then $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal. Thus, by Theorem 3.1, $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Let $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$. If $\mathfrak{u}\mathfrak{v} \in \mathcal{I}$, then $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}\mathfrak{v}) = \breve{1}$. By assumption, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}\mathfrak{v}) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v})$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v}) = \breve{1}$, so $\mathfrak{u} \in \mathcal{I}$ or $\mathfrak{v} \in \mathcal{I}$. Hence, $\mathcal{I}$ is a prime essential ideal of $\mathcal{S}$.

(3) Suppose that $\mathcal{I}$ is a semiprime essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Thus, by Theorem 4.1, $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal of $\mathcal{S}$. Let $\mathfrak{u} \in \mathcal{S}$.

If $\mathfrak{u}^2 \in \mathcal{I}$, then $\mathfrak{u} \in \mathcal{I}$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) = \breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) = \breve{1}$. Hence $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u})$.

If $\mathfrak{u}^2 \notin \mathcal{I}$, then $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) = 0 \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u})$.

Thus, $\breve{\chi}_{\mathcal{I}}$ is a semiprime essential IVF ideal of $\mathcal{S}$.

Conversely, suppose that $\breve{\chi}_{\mathcal{I}}$ is a semiprime essential IVF ideal of $\mathcal{S}$. Then $\breve{\chi}_{\mathcal{I}}$ is an essential IVF ideal. Thus, by Theorem 4.1, $\mathcal{I}$ is an essential ideal of $\mathcal{S}$. Let $\mathfrak{u} \in \mathcal{S}$ with $\mathfrak{u}^2 \in \mathcal{I}$. Then $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) = \breve{1}$. By assumption, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u})$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) = \breve{1}$, so $\mathfrak{u} \in \mathcal{I}$. Hence, $\mathcal{I}$ is a semiprime essential ideal of $\mathcal{S}$. □

**Theorem 3.6.** *Let $\breve{\eta}$ be a minimal essential IVF ideal of a semigroup $\mathcal{S}$. If $\breve{\nu}$ is an IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\nu}$, then $\breve{\nu}$ is also a minimal essential IVF ideal of $\mathcal{S}$.*

**Proof:** Let $\breve{\nu}$ be an IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\nu}$ and let $\breve{\mu}$ be any nonzero IVF ideal of $\mathcal{S}$ such that $\breve{\mu} \sqsubseteq \breve{\eta}$. Since $\breve{\eta}$ is a minimal essential IVF ideal of a semigroup $\mathcal{S}$, we have $\text{supp}(\breve{\eta}) = \text{supp}(\breve{\mu})$. Thus, $\text{supp}(\breve{\nu}) = \text{supp}(\breve{\mu})$. Hence, $\breve{\nu}$ is a minimal essential IVF ideal of $\mathcal{S}$. □

**Corollary 3.1.** *Let $\breve{\eta}_1$ and $\breve{\eta}_2$ be minimal essential IVF ideals of a semigroup $\mathcal{S}$. Then $\breve{\eta}_1 \sqcup \breve{\eta}_2$ is minimal essential fuzzy ideals of $\mathcal{S}$.*

4. **0-Essential IVF Ideal.** In this section, let $\mathcal{S}$ be a semigroup with zero. At the beginning, we review the definition 0-essential ideal of $\mathcal{S}$ as follows.

**Definition 4.1.** [11] *A nonzero ideal $\mathcal{I}$ of a semigroup with zero $\mathcal{S}$ is called a* 0-essential ideal *of $\mathcal{S}$ if $\mathcal{I} \cap \mathcal{J} \neq \{0\}$ for every nonzero ideal of $\mathcal{J}$ of $\mathcal{S}$.*

**Example 4.1.** [11] *Let $(\mathbb{Z}_{12}, +)$ be semigroup. Then $\{0, 2, 4, 6, 8, 10\}$ and $\mathbb{Z}_{12}$ are 0-essential ideal of $\mathbb{Z}_{12}$.*

**Definition 4.2.** *An IVF ideal $\breve{\eta}$ of a semigroup with zero $\mathcal{S}$ is called a* nontrivial IVF ideal *of $\mathcal{S}$ if there exists a nonzero element $\mathfrak{u} \in \mathcal{S}$ such that $\breve{\eta}(\mathfrak{u}) \neq \breve{0}$.*

We define the definition of 0-essential IVF ideals of a semigroup with zero as follows.

**Definition 4.3.** *A* 0-essential IVF ideal *$\breve{\eta}$ of a semigroup with zero $\mathcal{S}$ if $\breve{\eta}$ is a nonzero IVF ideal of $\mathcal{S}$ and $supp(\breve{\eta} \curlywedge \breve{\nu}) \neq \{0\}$ for every nonzero IVF ideal $\breve{\nu}$ of $\mathcal{S}$.*

**Theorem 4.1.** *Let $\mathcal{I}$ be a nonzero ideal of a semigroup with zero $\mathcal{S}$. Then $\mathcal{I}$ is a $0$-essential ideal of $\mathcal{S}$ if and only if $\breve{\chi}_{\mathcal{I}}$ is a $0$-essential IVF ideal of $\mathcal{S}$.*

**Proof:** Suppose that $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$ and let $\breve{\nu}$ be a nontrivial IVF ideal of $\mathcal{S}$. Then by Theorem 2.3, supp$(\breve{\nu})$ is a nonzero ideal of $\mathcal{S}$. Since $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$, we have $\mathcal{I}$ is a nonzero ideal of $\mathcal{S}$. Thus, $\mathcal{I} \cap \text{supp}(\breve{\nu}) \neq \{0\}$. So there exists $\mathfrak{u} \in \mathcal{I} \cap \text{supp}(\breve{\nu})$. Since $\mathcal{I}$ is a nonzero ideal of $\mathcal{S}$, we have $\breve{\chi}_{\mathcal{I}}$ is a nonzero IVF ideal of $\mathcal{S}$. Since $\breve{\nu}$ is a nonzero IVF ideal of $\mathcal{S}$, we have supp$(\breve{\chi}_{\mathcal{I}} \curlywedge \breve{\nu})(\mathfrak{u}) \neq 0$. Thus, $\breve{\chi}_{\mathcal{I}} \curlywedge \breve{\nu} \neq \breve{0}$. Therefore, $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal of $\mathcal{S}$.

Conversely, assume that $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be a nonzero ideal of $\mathcal{S}$. Then $\breve{\chi}_{\mathcal{J}}$ is a nonzero IVF ideal of $\mathcal{S}$. Sicne $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal of $\mathcal{S}$ we have $\breve{\chi}_{\mathcal{I}}$ is a nontrivial IVF ideal of $\mathcal{S}$. Thus, supp$(\breve{\chi}_{\mathcal{I}} \curlywedge \breve{\chi}_{\mathcal{J}}) \neq \{0\}$. So by Theorem 2.2, $\breve{\chi}_{\mathcal{L} \cap \mathcal{J}} \neq \breve{0}$. Hence, $\mathcal{I} \cap \mathcal{J} \neq \{0\}$. Therefore, $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. □

**Theorem 4.2.** *Let $\breve{\eta}$ be a nonzero IVF ideal of a semigroup with zero $\mathcal{S}$. Then $\breve{\eta}$ is a $0$-essential IVF ideal of $\mathcal{S}$ if and only if supp$(\breve{\eta})$ is a $0$-essential ideal of $\mathcal{S}$.*

**Proof:** Assume that $\breve{\eta}$ is a 0-essential IVF ideal of $\mathcal{S}$ and let $\mathcal{J}$ be a nontrivial ideal of $\mathcal{S}$. Then by Theorem 2.1, $\breve{\chi}_{\mathcal{J}}$ is a nonzero IVF ideal of $\mathcal{S}$. Since $\breve{\eta}$ is a 0-essential IVF ideal of $\mathcal{S}$, we have $\breve{\eta}$ is a nonzero IVF ideal of $\mathcal{S}$. Thus, $\breve{\eta} \curlywedge \breve{\chi}_{\mathcal{J}} \neq \breve{0}$. So there exists a nonzero element $\mathfrak{u} \in \mathcal{S}$ such that $(\breve{\eta} \curlywedge \breve{\chi}_{\mathcal{J}})(\mathfrak{u}) \neq \breve{0}$. It implies that $\breve{\eta}(\mathfrak{u}) \neq \breve{0}$ and $\breve{\chi}_{\mathcal{J}}(\mathfrak{u}) \neq \breve{0}$. Hence, $\mathfrak{u} \in \text{supp}(\breve{\eta}) \cap \mathcal{J}$, so supp$(\breve{\eta}) \cap \mathcal{J} \neq \{0\}$. Therefore, supp$(\breve{\eta})$ is a 0-essential ideal of $\mathcal{S}$.

Conversely, assume that supp$(\breve{\eta})$ is a 0-essential ideal of $\mathcal{S}$ and let $\breve{\nu}$ be a nonzero IVF ideal of $\mathcal{S}$. Then by Theorem 2.3, supp$(\breve{\nu})$ is a nontrivial zero ideal of $\mathcal{S}$. Since supp$(\breve{\eta})$ is a 0-essential ideal of $\mathcal{S}$, we have supp$(\breve{\eta})$ is a nonzero ideal of $\mathcal{S}$. Thus, supp$(\breve{\eta}) \cap \text{supp}(\breve{\nu})$ $\neq \breve{0}$. So there exists $\mathfrak{u} \in \text{supp}(\breve{\eta}) \cap \text{supp}(\breve{\nu})$, and this implies that $\breve{\eta}(\mathfrak{u}) \neq \breve{0}$ and $\breve{\nu}(\mathfrak{u}) \neq \breve{0}$ for all $\mathfrak{u} \in \mathcal{S}$. Hence, $(\breve{\eta} \curlywedge \breve{\nu})(\mathfrak{u}) \neq \breve{0}$ for all $\mathfrak{u} \in \mathcal{S}$. Therefore, $\breve{\eta} \curlywedge \breve{\nu} \neq \breve{0}$. We conclude that $\breve{\eta}$ is a 0-essential IVF ideal of $\mathcal{S}$. □

**Theorem 4.3.** *Let $\mathcal{S}$ be a semigroup with zero and $\breve{\eta}$ be a $0$-essential IVF ideal of $\mathcal{S}$. If $\breve{\nu}$ is an IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\nu}$, then $\breve{\nu}$ is also a $0$-essential IVF ideal of $\mathcal{S}$.*

**Proof:** Let $\breve{\nu}$ be an IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\nu}$ and let $\breve{\mu}$ be any nonzero IVF ideal of $\mathcal{S}$. Since $\breve{\eta}$ is a 0-essential IVF ideal of $\mathcal{S}$, we have $\breve{\eta}$ is an IVF ideal of $\mathcal{S}$. Thus, supp$(\breve{\eta} \curlywedge \breve{\mu}) \neq \{0\}$. So supp$(\breve{\nu} \curlywedge \breve{\mu}) \neq \{0\}$. Hence, $\breve{\nu}$ is a 0-essential IVF ideal of $\mathcal{S}$. □

**Theorem 4.4.** *Let $\breve{\eta}_1$ and $\breve{\eta}_2$ be $0$-essential IVF ideals of a semigroup with zero $\mathcal{S}$. Then $\breve{\eta}_1 \sqcup \breve{\eta}_2$ and $\breve{\eta}_1 \sqcap \breve{\eta}_2$ are $0$-essential IVF ideals of $\mathcal{S}$.*

**Proof:** Since $\breve{\eta}_2 \sqsubseteq \breve{\eta}_1 \sqcup \breve{\eta}_2$, we have $\breve{\eta}_1 \sqcup \breve{\eta}_2$ is a 0-essential IVF ideal of $\mathcal{S}$.

Since $\breve{\eta}_1$ and $\breve{\eta}_2$ are 0-essential IVF ideals of $\mathcal{S}$, we have $\breve{\eta}_1$ and $\breve{\eta}_2$ are IVF ideals of $\mathcal{S}$. Thus, $\breve{\eta}_1 \sqcap \breve{\eta}_2$ is an IVF ideal of $\mathcal{S}$. Let $\breve{\nu}$ be a nontrivial IVF ideal of $\mathcal{S}$. Since $\breve{\eta}_1$ is an IVF ideal of $S$, we have supp$(\breve{\eta}_1)$ is an ideal of $\mathcal{S}$. Thus, supp$(\breve{\eta}_1 \curlywedge \breve{\nu}) \neq \{0\}$. So there exists a nonzero element $\mathfrak{u} \in \mathcal{S}$ such that $(\breve{\eta}_1 \curlywedge \breve{\nu})(\mathfrak{u}) \neq \breve{0}$. Since $\breve{\eta}_2$ is a 0-essential IVF ideal of $\mathcal{S}$, we have supp$(\breve{\eta}_2)$ is a 0-essential IVF ideal of $\mathcal{S}$. Thus, supp$(\breve{\eta}_2 \curlywedge \breve{\nu}) \neq \{0\}$. So there exists a nonzero element $\mathfrak{v} \in \text{supp}(\breve{\eta}_2 \curlywedge \breve{\nu})$ that implies $(\breve{\eta}_2)(\mathfrak{v}) \neq \breve{0}$. Since $\breve{\eta}_1$ and $\breve{\nu}$ are IVF ideals of $\mathcal{S}$, we have $\breve{\eta}_1(\mathfrak{v}) \succeq \breve{\eta}_1(\mathfrak{u})$ and $\breve{\nu}(\mathfrak{v}) \succeq \breve{\nu}(\mathfrak{u})$. So $((\breve{\eta}_1 \curlywedge \breve{\eta}_2) \curlywedge \breve{\nu})(\mathfrak{v}) \neq \breve{0}$. Thus, supp$((\breve{\eta}_1 \curlywedge \breve{\eta}_2) \curlywedge \breve{\nu}) \neq \{0\}$. Therefore, $\breve{\eta}_1 \sqcap \breve{\eta}_2$ is a 0-essential IVF ideal of $\mathcal{S}$. □

**Definition 4.4.** [11] *A $0$-essential ideal $\mathcal{I}$ of a semigroup with zero $\mathcal{S}$ is called*

(1) *a* minimal *if for every $0$-essential ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{I}$, we have $\mathcal{J} = \mathcal{I}$,*
(2) *a* prime *if $\mathfrak{uv} \in \mathcal{I}$ implies $\mathfrak{u} \in \mathcal{I}$ or $\mathfrak{v} \in \mathcal{I}$,*
(3) *a* semiprime *if $\mathfrak{u}^2 \in \mathcal{I}$ implies $\mathfrak{u} \in \mathcal{I}$, for all $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.*

**Example 4.2.** *Let $(\mathbb{Z}_{12}, +)$ be a semigroup with zero. Then $\{0, 2, 4, 6, 8, 10\}$ is a minimal $0$-essential ideal of $\mathcal{S}$.*

**Definition 4.5.** *A* 0*-essential fuzzy ideal* $f$ *of a semigroup with zero* $\mathcal{S}$ *is called*

(1) *a* minimal *if for every* 0*-essential IVF ideal of* $\breve{\nu}$ *of* $\mathcal{S}$ *such that* $\breve{\nu} \subseteq \breve{\eta}$*, we have* $supp(\breve{\eta}) = supp(\breve{\nu})$*,*
(2) *a* prime *if* $\breve{\eta}(\mathfrak{uv}) \preceq \breve{\eta}(\mathfrak{u}) \curlyvee \breve{\eta}(\mathfrak{v})$*,*
(3) *a* semiprime *if* $\breve{\eta}(\mathfrak{u}^2) \preceq \breve{\eta}(\mathfrak{u})$*, for all* $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$*.*

**Theorem 4.5.** *Let* $\mathcal{I}$ *be a non-empty subset of a semigroup with zero* $\mathcal{S}$*. Then the following statements hold.*

(1) $\mathcal{I}$ *is a minimal* 0*-essential ideal of* $\mathcal{S}$ *if and only if* $\breve{\chi}_{\mathcal{I}}$ *is a minimal* 0*-essential IVF ideal of* $\mathcal{S}$*,*
(2) $\mathcal{I}$ *is a prime* 0*-essential ideal of* $\mathcal{S}$ *if and only if* $\breve{\chi}_{\mathcal{I}}$ *is a prime* 0*-essential IVF ideal of* $\mathcal{S}$*,*
(3) $\mathcal{I}$ *is a semiprime* 0*-essential ideal of* $S$ *if and only if* $\breve{\chi}_{\mathcal{I}}$ *is a semiprime* 0*-essential IVF ideal of* $\mathcal{S}$*.*

**Proof:**

(1) Suppose that $\mathcal{I}$ is a minimal 0-essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. By Theorem 4.1, $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal of $\mathcal{S}$. Let $\breve{\eta}$ be a 0-essential IVF ideal of $\mathcal{S}$ such that $\breve{\eta} \sqsubseteq \breve{\chi}_{\mathcal{I}}$. Then $\text{supp}(\breve{\eta}) \sqsubseteq \text{supp}(\breve{\chi}_{\mathcal{I}})$. Thus, $\text{supp}(\breve{\eta}) \sqsubseteq \text{supp}(\breve{\chi}_{\mathcal{I}}) = \mathcal{I}$. Thus, $\text{supp}(\breve{\eta}) \sqsubseteq \mathcal{I}$. Since $\breve{\eta}$ is a 0-essential IVF ideal of $\mathcal{S}$, we have $\text{supp}(\breve{\eta})$ is 0-essential ideal of $\mathcal{S}$. By assumption, $\text{supp}(\breve{\eta}) = \mathcal{I} = \text{supp}(\breve{\chi}_{\mathcal{I}})$. Hence, $\breve{\chi}_{\mathcal{I}}$ is a minimal 0-essential IVF ideal of $\mathcal{S}$.

Conversely, $\breve{\chi}_{\mathcal{I}}$ is a minimal 0-essential IVF ideal of $\mathcal{S}$ and let $\mathcal{B}$ be a 0-essential ideal of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{I}$. Then $\mathcal{B}$ is a nonzero ideal of $\mathcal{S}$. Thus, by Theorem 4.1, $\breve{\chi}_{\mathcal{B}}$ is a 0-essential IVF ideal of $\mathcal{S}$ such that $\breve{\chi}_{\mathcal{B}} \sqsubseteq \breve{\chi}_{\mathcal{I}}$. So $\breve{\chi}_{\mathcal{B}} = \breve{\chi}_{\mathcal{I}}$. Hence, $\mathcal{B} = \text{supp}(\breve{\chi}_{\mathcal{B}}) = \text{supp}(\breve{\chi}_{\mathcal{I}}) = \mathcal{I}$. Therefore, $\mathcal{I}$ is a minimal 0-essential ideal of $\mathcal{S}$.

(2) Suppose that $\mathcal{I}$ is a prime 0-essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Thus, by Theorem 3.1, $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal of $\mathcal{S}$. Let $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$.

If $\mathfrak{uv} \in \mathcal{I}$, then $\mathfrak{u} \in \mathcal{I}$ or $\mathfrak{v} \in \mathcal{I}$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v}) = \breve{1} \succeq \breve{\chi}_{\mathcal{I}}(\mathfrak{uv})$.

If $\mathfrak{uv} \notin \mathcal{I}$, then $\breve{\chi}_{\mathcal{I}}(\mathfrak{uv}) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v})$.

Thus, $\breve{\chi}_{\mathcal{I}}$ is a prime 0-essential IVF ideal of $\mathcal{S}$.

Conversely, suppose that $\breve{\chi}_{\mathcal{I}}$ is a prime 0-essential IVF ideal of $\mathcal{S}$. Then $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal. Thus, by Theorem 4.1, $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Let $\mathfrak{u}, \mathfrak{v} \in \mathcal{S}$. If $\mathfrak{uv} \in \mathcal{I}$, then $\breve{\chi}_{\mathcal{I}}(\mathfrak{uv}) = \breve{1}$. By assumption $\breve{\chi}_{\mathcal{I}}(\mathfrak{uv}) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v})$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) \curlyvee \breve{\chi}_{\mathcal{I}}(\mathfrak{v}) = \breve{1}$ so $\mathfrak{u} \in \mathcal{I}$ or $\mathfrak{v} \in \mathcal{I}$. Hence, $\mathcal{I}$ is a prime 0-essential ideal of $\mathcal{I}$.

(3) Suppose that $\mathcal{I}$ is a semiprime 0-essential ideal of $\mathcal{S}$. Then $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Thus, by Theorem 4.1, $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal of $\mathcal{S}$. Let $\mathfrak{u} \in \mathcal{S}$.

If $\mathfrak{u}^2 \in \mathcal{I}$, then $\mathfrak{u} \in \mathcal{I}$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) = \breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) = \breve{1}$. Hence, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u})$.

If $\mathfrak{u}^2 \notin \mathcal{I}$, then $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) = 0 \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u})$.

Thus, $\breve{\chi}_{\mathcal{I}}$ is a semiprime 0-essential IVF ideal of $\mathcal{S}$.

Conversely, suppose that $\breve{\chi}_{\mathcal{I}}$ is a semiprime 0-essential IVF ideal of $\mathcal{S}$. Then $\breve{\chi}_{\mathcal{I}}$ is a 0-essential IVF ideal. Thus, by Theorem 4.1, $\mathcal{I}$ is a 0-essential ideal of $\mathcal{S}$. Let $\mathfrak{u} \in \mathcal{S}$ with $\mathfrak{u}^2 \in \mathcal{I}$. Then $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) = \breve{1}$. By assumption, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}^2) \preceq \breve{\chi}_{\mathcal{I}}(\mathfrak{u})$. Thus, $\breve{\chi}_{\mathcal{I}}(\mathfrak{u}) = \breve{1}$, so $\mathfrak{u} \in \mathcal{I}$. Hence, $\mathcal{I}$ is a semiprime 0-essential ideal of $\mathcal{S}$. □

## 5. Conclusion.

In this article, we give the concept of essential IVF ideals and 0-essential IVF ideals in semigroups. We study properites of essential IVF ideals and 0-essential IVF ideals in semigroups. In the future, we study essential hesitant fuzzy ideals and 0-essential hesitant fuzzy ideals in semigroups or algebraic systems.

**Acknowledgment.** This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF66-RIM024).

The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

## REFERENCES

[1] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol.8, pp.338-353, 1965.
[2] N. Kuroki, Fuzzy bi-ideals in semigroup, *Comment. Math. Univ. St. Paul.*, vol.5, pp.128-132, 1979.
[3] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, *Information Sciences*, vol.8, pp.199-249, 1975.
[4] H. Bustince, Indicator of inclusion grade for interval valued fuzzy sets. Application to approximate reasoning based on interval valued fuzzy sets, *International Journal of Approximate Reasoning*, vol.23, pp.137-209, 1998.
[5] J. Aranzazu, S. Antonio, P. Daniel, F. Javier and B. Humberto, Interval valued fuzzy sets for color image super-resolution, *Advances in Artificial Intelligence*, pp.373-382, 2011.
[6] M. Zulquanain and M. Saeed, A new decision making method on interval value fuzzy soft matrix, *British Journal of Mathematics and Computer Science*, vol.20, no.5, pp.1-17, 2017.
[7] A. L. Narayanan and T. Manikantan, Interval valued fuzzy ideals generated by an interval valued fuzzy subset in semigroups, *Journal of Applied Mathematics and Computing*, vol.20, nos.1-2, pp.455-464, 2006.
[8] U. Medhi, K. Rajkhowa, L. K. Barthakur and H. K. Saikia, On fuzzy essential ideals of ring, *Advances in Fuzzy Sets and Systems*, vol.3, pp.287-299, 2008.
[9] U. Medhi and H. K. Saikia, On T-fuzzy essential ideals of rings, *Journal of Pure and Applied Mathematics*, vol.89, pp.343-353, 2013.
[10] S. Wani and K. Pawar, On essential ideals of a ternary semiring, *Sohag Journal of Mathematics*, vol.4, no.3, pp.65-68, 2017.
[11] S. Baupradist, B. Chemat, K. Palanivel and R. Chinram, Essential ideals and essential fuzzy ideals in semigroups, *Journal of Discrete Mathematical Sciences and Cryptography*, vol.24, no.1, pp.223-233, 2021.
[12] R. Chinram and T. Gaketem, Essential $(m, n)$-ideal and essential fuzzy $(m, n)$-ideals in semigroups, *ICIC Express Letters*, vol.15, no.10, pp.1037-1044, 2021.
[13] N. Panpetch, T. Muangngao and T. Gaketem, Some essential bi-ideals and essential fuzzy bi-ideals in a semigroup, *Journal of Mathematics and Computer Science*, vol.28, no.4, pp.326-334, 2023.
[14] T. Gaketem and A. Iampan, Essential UP-ideals and $t$-essential fuzzy UP-ideals of UP-algebra, *ICIC Express Letters*, vol.15, no.12, pp.1283-1289, 2021.
[15] T. Gaketem, P. Khamrot and A. Iampan, Essential UP-filters and $t$-essential fuzzy UP-filters of UP-algebra, *ICIC Express Letters*, vol.16, no.10, pp.1057-1062, 2022.
[16] R. Rittichuai, A. Iampan, R. Chinram and P. Singavananda, Essential ideals and their fuzzifications of ternary semigroups, *ICIC Express Letters*, vol.17, no.2, pp.191-199, 2023.
[17] J. N. Mordeson, D. S. Malik and N. Kuroki, *Fuzzy Semigroup*, Springer Science and Business Media, 2003.