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ON THE MODULES HAVING PROPER S-ESSENTIAL SUBMODULES

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Abstract. *In this paper, we study S-essential submodules which are a generalization of essential submodule of modules. Besides giving many examples and properties of S-essential submodules, we generalize some results on essential submodules to S-essential submodules. Finally, we can show that for a nonzero right R-module M such that $Ann_R(m) \cap S = \varnothing$ for all $m \in M - \{0\}$, M has a proper S-essential submodule if and only if a right $R/Ann_R(M)$-module M has a proper $\overline{S}$-essential submodule where $\overline{S} = \{s + Ann_R(M) | s \in S\}$.*

Keywords: Multiplicatively closed subset, Essential submodule, S-essential submodule, Essential monomorphism, S-essential monomorphism

1. **Introduction and Preliminaries.** Throughout this paper, R is an associative ring with identity and all R-modules will be unitary right R-modules. The annihilator of right R-module M is denoted by $Ann_R(M)$ where $Ann_R(M) = \{r \in R | Mr = 0\}$. From [1], a nonempty subset S of R is called a *multiplicatively closed subset* (briefly, m.c.s.) of R, if $0 \notin S$, $1 \in S$ and $ss' \in S$ for all $s, s' \in S$. In the sequel, unless stated otherwise, S is always a multiplicatively closed subset of ring R. From [2], a proper submodule N of a right R-module M is called an *essential submodule* in M, if $N \cap K \neq 0$ for all non-zero submodule K of M, or equvalently, for submodule K of M such that $N \cap K = 0$ implies that $K = 0$ and we write $N \ll_e M$. Many authors have been interested in studying defferent definitions generalization of essential submodules (see [3, 4, 5, 6]). Recently, in [7], Rajaee introduced the notion of S-essential submodule which is a generalization of essential submodule. A submodule N of right R-module M is called *S-essential* (*S-large*) submodule of M and denote by $N \ll_e^S M$, if for every submodule L of M the equality $N \cap L = 0$ implies that there exists an $s \in S$ such that $Ls = 0$. He generalized the concepts of essential submodule of a right R-module M to the S-essential submodule of M where S is a multiplicatively closed subset of R and provided some useful theorem concerning this new class of submodules. For the basic concepts and other notations, we refer the readers to [1, 2, 7, 8].

In this paper, we continue the work of Rajaee [7] and give many other examples. In Section 2, several properties of these classes of submodules are considered. In Section 3, we consider some conditions that a right R-module has a proper S-essential submodule. In Section 4, we conclude this paper with future work.

Let us begin by giving examples of S-essential submodule of right R-module and some remark of this concept.

Example 1.1. (1) *Every essential submodule of right R-module M is an S-essential submodule of M for all multiplicatively closed subset S of R.*

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(2) *The converse of (1) is not true in general. For example, let Z be the set of all integers and Z_6 be the set of all integers modulo 6.*

(2.1) *Consider Z_6 as a right Z-module and $S = \{1, 3, 6, 9, \ldots\}$. Then S is a multiplicatively closed subset of Z. Let U be a submodule of Z_6 such that $\langle\overline{3}\rangle \cap U = 0$. Since $\langle\overline{0}\rangle$, $\langle\overline{1}\rangle = \langle\overline{5}\rangle$, $\langle\overline{2}\rangle = \langle\overline{4}\rangle$ and $\langle\overline{3}\rangle$ are of all submodules of Z_6, $U = 0$ or $U = \langle\overline{2}\rangle$.*
Case 1. $U = 0$. Choose $s \in S$. We have $Us = 0$.
Case 2. $U = \langle\overline{2}\rangle$. There is $3 \in S$ and $\langle\overline{2}\rangle 3 = 0$.
Hence, $\langle\overline{3}\rangle$ is an S-essential submodule of Z_6. However, $\langle\overline{3}\rangle$ is not an essential submodule of Z_6. Since $\langle\overline{3}\rangle \cap \langle\overline{2}\rangle = 0$, $\langle\overline{2}\rangle \neq 0$.

(2.2) *Consider Z_6 as right Z_6-module and $S = \{\overline{1}, \overline{3}\}$. Then S is a multiplicatively closed subset of Z_6. Let U be a submodule of Z_6 such that $\langle\overline{3}\rangle \cap U = 0$. Then $U = 0$ or $U = \langle\overline{2}\rangle$.*
Case 1. $U = 0$. There is $\overline{1} \in S$ and $0\overline{1} = 0$.
Case 2. $U = \langle\overline{2}\rangle$. There is $\overline{3} \in S$ and $\langle\overline{2}\rangle \overline{3} = 0$.
From case 1 and case 2, $\langle\overline{3}\rangle \ll_e^S Z_6$. Even in this example, we consider Z_6 as right Z_6-module. We also concluded that $\langle\overline{3}\rangle$ is not an essential submodule of Z_6.

(2.3) *From (2.2), we have 0 is not an S-essential submodule of Z_6.*

(3) *The converse of (1) is true provided that $S \subseteq U(R)$ where $U(R)$ is the set of all units in R.*

(4) *In the case that $S = \{1\}$, every S-essential submodule of right R-module is an essential submodule.*

(5) *If M is a uniserial module then for any nonzero submodule of M is an S-essential submodule for each multiplicatively closed subset S of R.*

Remark 1.1. *Let M be a right R-module.*

(1) $M \ll_e^S M$.

(2) *There exists a right R-module M such that for any submodule U of M, U is an S-essential submodule (In the future, we will refer to this concept as S-Uniform modules). We consider, Z_n is a right Z-module and $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha 3} \ldots p_n^{\alpha_n}$ where p_i are prime numbers and α_i are positive integers for all $i = 1, 2, 3, \ldots, n$. Let $p = \min\{p_i | i = 1, 2, 3, \ldots, n\}$ and $S = \{1, p, 2p, 3p, \ldots\}$. Then S is a multiplicatively closed subset of Z. We know that every submodule of Z_n is the form $\langle\overline{a}\rangle$ where a is a positive integer such that $a|n$. By definition of S, we have $n \in S$ and for all submodule U of Z_n, $Un = 0$. So for any submodule of Z_n is an S-essential submodule of Z_n. Hence, 0 is an S-essential submodule of Z_n.*

(3) *If 0 is an S-essential submodule of right R-module M then for any submodule U of M there exists $s \in S$ such that $Us = 0$.*

By the above example, we can conclude that $\{0\}$ is not an essential submodule of any right R-module M. However, depending on the multiplicatively closed subset S of a ring R, $\{0\}$ will be an S-essential submodule of right R-module M.

2. *S*-Essential Submodules.

In this section, we give an S-essential submodule concept as a generalization of essential submodule. Also, we generalize some properties of essential submodules to S-essential submodules.

Proposition 2.1. *Let A, B and C be submodules of M. If $A \ll B \ll C \ll M$ and $A \ll_e^S M$, then $B \ll_e^S C$.*

Proof: Suppose that $A \ll B \ll C \ll M$ and $A \ll_e^S M$. Let $L \ll C$ such that $B \cap L = 0$. Since $A \ll B$, $A \cap L = 0$. However, $A \ll_e^S M$, there is an $s \in S$ such that $Ls = 0$. Hence, $B \ll_e^S C$. □

Corollary 2.1. *Let N_1 and N_2 be submodules of a right R-module M. If $N_1 \cap N_2$ is an S-essential submodule of M, then N_1 and N_2 are S-essential submodules of M.*

Proof: By Proposition 2.1. □

Proposition 2.2. *Let M be a right R-module. If $A \ll_e B \ll M$ and $A_1 \ll_e^S B_1 \ll M$, then $A \cap A_1 \ll_e^S B \cap B_1$.*

Proof: Suppose that $A \ll_e B \ll M$ and $A_1 \ll_e^S B_1 \ll M$. Let L be a submodule of $B \cap B_1$ such that $(A \cap A_1) \cap L = 0$. Then $A \cap (A_1 \cap L) = 0$. Since $A \ll_e B$ and $A_1 \cap L \ll L \ll B$, $A_1 \cap L = 0$. However, $A_1 \ll_e^S B_1$ and $L \ll B_1$, there exists $s \in S$ such that $Ls = 0$. Hence, $A \cap A_1 \ll_e^S B \cap B_1$. □

Proposition 2.3. *Let A and B be submodules of a right R-module M. If $A \ll_e^S B$, then $A \cap C \ll_e^S B \cap C$ for all submodule C of M.*

Proof: Suppose that $A \ll_e^S B$. Let C be a submodule of M and D a submodule of $B \cap C$ such that $(A \cap C) \cap D = 0$. Then $A \cap (C \cap D) = 0$ but $A \ll_e^S B$, there is $s \in S$ such that $(C \cap D)s = 0$. Since $D \subseteq C$, $Ds = 0$. Therefore, $A \cap C \ll_e^S B \cap C$. □

Theorem 2.1. *Let M be a right R-module such that $Ann_R(m) \cap S = \varnothing$ for all $m \in M - \{0\}$ and L_1, L_2, K_1, K_2 be submodules of M. If $K_1 \ll_e^S L_1$ and $K_2 \ll_e^S L_2$, then $K_1 \cap K_2 \ll_e^S L_1 \cap L_2$.*

Proof: Let X be a submodule of $L_1 \cap L_2$. Suppose that $Xs \neq 0$ for all $s \in S$. Since $X \ll L_1$ and $K_1 \ll_e^S L_1$, $K_1 \cap X \neq 0$. Assume that there exists $s \in S$ such that $(X \cap K_1)s = 0$. Since $K_1 \cap X \neq 0$, there exists $0 \neq a \in K_1 \cap X$, and thus, $as \in (K_1 \cap X)s = 0$. So $as = 0$. This is a contradiction with $Ann_R(a) \cap S = \varnothing$. So $(K_1 \cap X)s \neq 0$ for all $s \in S$. Since $X \cap K_1 \ll X \ll L_1 \cap L_2 \ll L_2$ and $K_2 \ll_e^S L_2$, $X \cap (K_1 \cap K_2) = (X \cap K_1) \cap K_2 \neq 0$. Therefore, $K_1 \cap K_2 \ll_e^S L_1 \cap L_2$. □

Proposition 2.4. *Let $M = M_1 \oplus M_2$ be a right R-module, K_1 a submodule of M_1 and K_2 submodule of M_2. If $K_1 \oplus K_2 \ll_e^S M_1 \oplus M_2$, then $K_1 \ll_e^S M_1$ and $K_2 \ll_e^S M_2$.*

Proof: Suppose that K_1 is not an S-essential submodule of M_1 or K_2 is not an S-essential submodule of M_2.

Case 1. K_1 is not an S-essential submodule of M_1. There exists a submodule L_1 of M_1 such that $L_1 \cap K_1 = 0$ but $L_1 s \neq 0$ for all $s \in S$. Let $l \in (K_1 + K_2) \cap L_1$. Then $l \in L_1$ and $l = k_1 + k_2$ for some $k_1 \in K_1$ and $k_2 \in K_2$. So $k_2 = l - k_1 \in M_1$ and we have $k_2 \in M_1 \cap M_2 = 0$. Then $k_2 = 0$, and thus, $l = k_1$. Then $k_1 \in K_1 \cap L_1 = 0$. This implies that $l = k_1 + k_2 = 0$. Hence, $(K_1 + K_2) \cap L_1 = 0$ but $L_1 s \neq 0$ for all $s \in S$. It concludes that $K_1 \oplus K_2$ is not an S-essential submodule of M.

Case 2. K_2 is not an S-essential submodule of M_2. This case is similar to case 1.

From case 1 and case 2, we can conclude that $K_1 \oplus K_2$ is not an S-essential submodule of M. □

Lemma 2.1. *Let M be a right R-module and K a submodule of M. If for each $0 \neq m \in M$ there exists $r \in R$ such that $0 \neq mr \in K$, then K is an S-essential submodule of M.*

Proof: Let L be a submodule of M such that $Ls \neq 0$ for all $s \in S$. Since $1 \in S$, $L \neq 0$. We can choose $l \in L - \{0\}$. By assumption, there exixts $r \in R$ such that $lr \in K$. So $0 \neq lr \in K \cap L$. That is $K \cap L \neq 0$. Therefore, K is an S-essential submodule of M. □

Lemma 2.2. *Let M be a right R-module and K an S-essential submodule of M. For any $m \in M$, if $Ann_R(m) \cap S = \varnothing$, then there exists $r \in R$ such that $0 \neq mr \in K$.*

Proof: Let $m \in M$ such that $Ann_R(m) \cap S = \varnothing$. Then $mRs \neq 0$ for all $s \in S$. However, K is an S-essential submodule of M, $K \cap mR \neq 0$. There exists $r \in R$ such that $0 \neq mr \in K$. □

Theorem 2.2. *Let $M = M_1 \oplus M_2$ be a right R-module, K_1 a submodule of M_1 and K_2 a submodule of M_2 such that for each $m \in M_i - \{0\}$, $Ann_R(m) \cap S = \varnothing$ where $i = 1, 2$. If $K_1 \ll_e^S M_1$ and $K_2 \ll_e^S M_2$, then $K_1 \oplus K_2 \ll_e^S M_1 \oplus M_2$.*

Proof: Suppose that $K_1 \ll_e^S M_1$ and $K_2 \ll_e^S M_2$. Let $0 \neq m \in M = M_1 \oplus M_2$. Then $m = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$.

Case 1. There exists $i \in \{1, 2\}$ such that $m_i = 0$.

Subcase 1.1. $m_1 = 0$. Then $m_2 \neq 0$. By Proposition 2.2, there exists $r \in R$ such that $0 \neq m_2 r \in K_2$. Then $(m_1 + m_2)r = m_1 r + m_2 r = m_2 r \in K_1 \oplus K_2$. By Proposition 2.1, $K_1 \oplus K_2$ is an S-essential submodule of M.

Subcase 1.2. $m_2 = 0$. It is similar to subcase 1.1.

Case 2. $m_i \neq 0$ for all $i \in \{1, 2\}$. Since $K_1 \ll_e^S M_1$ and by Proposition 2.2, there exists $r_1 \in R$ such that $0 \neq m_1 r_1 \in K_1$.

Subcase 2.1. $m_2 r_1 \in K_2$. Then $m_1 r_1 + m_2 r_1 \neq 0$ and $(m_1 + m_2)r_1 = m_1 r_1 + m_2 r_1 \in K_1 \oplus K_2$.

Subcase 2.2. $m_2 r_1 \notin K_2$. It is obvious that $m_2 r_1 \neq 0$. Since $K_2 \ll_e^S M_2$ and by Proposition 2.2, there exists $r_2 \in R$ such that $0 \neq m_2 r_1 r_2 \in K_2$. Then $m_1 r_1 r_2 + m_2 r_1 r_2 \neq 0$ and $(m_1 + m_2) r_1 r_2 = m_1 r_1 r_2 + m_2 r_1 r_2 \in K_1 \oplus K_2$. From case 1, case 2 and by Proposition 2.1, we have $K_1 \oplus K_2$ is an S-essential of M. □

Proposition 2.5. *Let M and N be right R-modules and $f : M \longrightarrow N$ an isomorphism. If B is an S-essential submodule of M, then $f(B)$ is an S-essential submodule of N.*

Proof: Suppose that B is an S-essential submodule of M. Let L be a submodule of N such that $f(B) \cap L = 0$. Since f is a monomorphism, $B \cap f^{-1}(L) = 0$. However, B is an S-essential submodule of M, there exists $s \in S$ such that $f^{-1}(L)s = 0$. Suppose that $Ls \neq 0$. There exists $l \in L$ such that $ls \neq 0$. Since f is an epimorphism, there exists $m \in M$ such that $f(m) = l$. Then $m \in f^{-1}(L)$, and thus, $ms = 0$. We have $0 = f(ms) = f(m)s = ls$. This is a contradiction. Hence, $Ls = 0$. Therefore, $f(B)$ is an S-essential submodule of N. □

Proposition 2.6. *Let M, N be right R-modules over commutative ring R, A a submodule of N and $\varphi \in Hom_R(M, N)$. If $A \ll_e^S N$, then $\varphi^{-1}(A) \ll_e^S M$.*

Proof: Suppose that $A \ll_e^S N$. Let U be a submodule of M such that $\varphi^{-1}(A) \cap U = 0$. Then $A \cap \varphi(U) = 0$. By assumption, there is $s \in S$ such that $\varphi(U)s = 0$. Thus, $\varphi(Us) = 0$. However, R is a commutative ring, we have $Us \ll Ker(\varphi) = \varphi^{-1}(0) \ll \varphi^{-1}(A)$ and hence $Us = Us \cap \varphi^{-1}(A) \ll U \cap \varphi^{-1}(A) = 0$. Then $Us = 0$. Therefore, $\varphi^{-1}(A) \ll_e^S M$. □

Theorem 2.3. *Let M be a right R-module. Then the following statements are equivalent.*

(i) $0 \ll_e^S M$.

(ii) *For each submodule L of M, $L \ll_e^S M$.*

(iii) *For each proper submodule L of M, $L \ll_e^S M$.*

(iv) *For each submodule L of M, $L' \ll_e^S M$ where L' is a complement of L in M.*

(v) *For each submodule L of M, there exists $s \in S$ such that $Ls = 0$.*

Proof: (i) $\Longrightarrow$ (ii) By Remark 1.1 (3).

(ii) $\Longrightarrow$ (i) It is obvious.

(ii) $\Longrightarrow$ (iii) This is clear.

(iii) $\Longrightarrow$ (ii) By Remark 1.1 (1).

(i) $\Longrightarrow$ (iv) Let L be a submodule of M, L' a complement of L in M and K a submodule of M such that $L' \cap K = 0$. By Remark 1.1 (3), there exists $s \in S$ such that $Ks = 0$. Hence, L' is an S-essential submodule of M.

(iv) $\Longrightarrow$ (i) Let K be a submodule of M and K' a complement of K in M. By assumption, K' is an S-essential submodule of M. However, $K \cap K' = 0$, there exists $s \in S$ such that $Ks = 0$. Hence, 0 is an S-essential submodule of M.

(i) $\Longrightarrow$ (v) By Remark 1.1 (3).
(v) $\Longrightarrow$ (i) This is clear. □

Corollary 2.2. *Let M be a right R-module and L a submodule of M. If a complement L' of L is an S-essential submodule of M, then there exists $s \in S$ such that $Ls = 0$.*

Proof: Similar in the proof of Theorem 2.3 ((iv) $\Longrightarrow$ (i)). □

Definition 2.1. *Let M, N be right R-modules and S a multiplicatively closed subset of a ring R. A monomorphism $f : M \longrightarrow N$ is said to be S-essential monomorphism, if $Im(f)$ is an S-essential submodule of N.*

Recall that an R-homomorphism $f : M \longrightarrow N$ of R-modules M and N is said to be *essential monomorphism* [10], $Im(f)$ is an essential submodule of N.

Example 2.1. *Every essential monomorphism from M to N is an S-essential monomorphism from M to N where M and N are right R-modules.*

Proposition 2.7. *Let M and N be right R-modules. If $S \subseteq U(R)$, then every S-essential monomorphism from M to N is an essential monomorphism from M to N where $U(R)$ is the set of all units in R.*

Proof: This is clear. □

Theorem 2.4. *Let M be a right R-module and K a submodule of M. Then the following statements are equivalent.*

(i) *K is an S-essential submodule of M.*
(ii) *The inclusion map $i_K : K \longrightarrow M$ is an S-essential monomorphism.*
(iii) *For each right R-module N and $f \in Hom_R(M, N)$, if $(Ker(f)) \cap K = 0$, then there exists $s \in S$ such that $(Ker(f))s = 0$.*

Proof: (i) $\Longleftrightarrow$ (ii) It is obvious.
(i) $\Longrightarrow$ (iii) This is clear.
(iii) $\Longrightarrow$ (i) Let L be a submodule of M such that $K \cap L = 0$ and η_L a natural map from M to M/L. Then $\eta_L \in Hom_R(M, M/L)$ and $Ker(\eta_L) = L$, and thus, $Ker(\eta_L) \cap K = 0$. By assumption, there exists $s \in S$ such that $(Ker(\eta_L))s = 0$. So $Ls = 0$. Hence, K is an S-essential submodule of M. □

Theorem 2.5. *Let L and M be right R-modules and f a monomorphism from L to M. Then f is S-essential if and only if for each right R-module N and for each R-homomorphism $h : M \longrightarrow N$ such that $h \circ f$ is a monomorphism there exists $s \in S$ such that $(Ker(h))s = 0$.*

Proof: ($\Longrightarrow$) Let $h : M \longrightarrow N$ such that $h \circ f$ is monomorphism. Let $x \in Ker(h \circ f)$. Then $h(f(x)) = 0$ and thus $f(x) \in Ker(h) \cap Im(f)$. So $x \in f^{-1}(Ker(h) \cap Im(f))$. We have $Ker(h \circ f) \subseteq f^{-1}(Ker(h) \cap Im(f))$. Conversely, let $x \in f^{-1}(Ker(h) \cap Im(f))$. Then $f(x) \in Ker(h)$ and thus $h(f(x)) = 0$. So $x \in Ker(h \circ f)$. Hence, $f^{-1}(Ker(h) \cap Im(f)) = Ker(h \circ f)$. However, $h \circ f$ is a monomorphism, $f^{-1}(Ker(h) \cap Im(f)) = 0$. Let $y \in Ker(h) \cap Im(f)$. Then $h(y) = 0$ and $f(x) = y$ for some $x \in L$. Since $0 = h(y) = h(f(x))$, $f(x) \in Ker(h) \cap Im(f)$. So $x \in f^{-1}(Ker(h) \cap Im(f)) = 0$ and thus $y = f(x) = f(0) = 0$. We have $Ker(h) \cap Im(f) = 0$ but $Im(f) \ll_e^S M$, there exists $s \in S$ such that $(Ker(h))s = 0$.

($\Longleftarrow$) Let K be a submodule of M such that $Im(f) \cap K = 0$. Let π be a canonical projection map from M to M/K. Let $x \in Ker(\pi \circ f)$. Then $\pi(f(x)) = 0$. So $f(x) + K = K$ and thus $f(x) \in K$. We have $f(x) \in Im(f) \cap K = 0$. That is $f(x) = 0$ but f is a monomorphism, $x = 0$. So $\pi \circ f$ is a monomorphism and by assumption, there exists $s \in S$ such that $(Ker(\pi))s = 0$. Since $Ker(\pi) = K$, $Ks = 0$. Hence, f is an S-essential monomorphism. □

Corollary 2.3. *Let L, M, N be right R-modules and $f : L \longrightarrow M$, $g : M \longrightarrow N$ be monomorphisms. If $g \circ f$ is an S-essential monomorphism, then g is an S-essential monomorphism.*

Proof: Suppose that $g \circ f$ is an S-essential monomorphism. Let M' be a right R-module and $h : M \longrightarrow M'$ an R-homomorphism such that $h \circ g$ is a monomorphism. Then $h \circ (g \circ f)$ is a monomorphism. Since $g \circ f$ is an S-essential and by Theorem 2.5, there exists $s \in S$ such that $(Ker(h))s = 0$. By Theorem 2.5, g is an S-essential monomorphism. □

Theorem 2.6. *Let L, M, N be right R-modules and $f : L \longrightarrow M$, $g : M \longrightarrow N$ be monomorphisms. If f is an epimorphism and g is an S-essential, then $g \circ f$ is an S-essential.*

Proof: Let N' be a right R-module and $h : N \longrightarrow N'$ such that $h \circ g \circ f$ is a monomorphism. Then $Ker(h \circ g \circ f) = 0$. Let $m \in Ker(h \circ g)$. So $h(g(m)) = 0$. Since f is an epimorphism, there exists $l \in L$ such that $f(l) = m$. However, $0 = h(g(m)) = h(g(f(l)))$, $l \in Ker(h \circ g \circ f) = 0$ and thus $l = 0$. We have $m = f(l) = f(0) = 0$. Thus, $Ker(h \circ g) = 0$. Since g is S-essential and by Theorem 2.5, there exists $s \in S$ such that $(Ker(h))s = 0$. By Theorem 2.5, $g \circ f$ is an essential. □

From [9], a right R-module M is called *multiplication module*, if for every submodule N of M, there exists an ideal I of R such that $N = MI$.

Theorem 2.7. *Let M be a faithful multiplication module and N a submodule of M such that $N = MI$ for some ideal I of R. If I is an S-essential ideal of R, then N is an S-essential submodule of M.*

Proof: Suppose that I is an S-essential ideal of R. Let L be a submodule of M such that $N \cap L = 0$. Since M is a multiplication module, there exists ideal J of R such that $L = MJ$. Then $MI \cap MJ = N \cap L = 0$ but $M(I \cap J) \subseteq MI \cap MJ$ and thus $M(I \cap J) = 0$. Since M is a faithful module, $I \cap J = 0$. By assumption, there exists $s \in S$ such that $Js = 0$. So $Ls = (MJ)s = M(Js) = M0 = 0$. Hence, N is an S-essential submodule of M. □

3. **Proper *S*-Essential Submodules.** We finish this paper by finding some conditions that a right R-module has a proper S-essential submodule.

Example 3.1.

(i) *Consider Z_p as a right Z_p-module where p is a prime number. Then Z_p has not proper S-essential submodule for any multiplicatively closed subset S of Z_p.*
(ii) *Consider Z_{pq} as a right Z_{pq}-module where p and q are distinct prime numbers. Let S be the set of units of ring R. Then S is a multiplicatively closed subset of Z_{pq}. Then Z_{pq} has not proper S-essential submodule.*
(iii) *Consider Z_{12} as a right Z_{12}-module. We have $\langle \overline{2} \rangle$, $\langle \overline{3} \rangle$ and $\langle \overline{6} \rangle$ are of all proper S-essential submodules of Z_{12} where $S = \{\overline{1}, \overline{3}, \overline{9}\}$.*
(iv) *Consider Z_{24} as a right Z_{24}-module. We have $\langle \overline{2} \rangle$, $\langle \overline{3} \rangle$, $\langle \overline{4} \rangle$, $\langle \overline{6} \rangle$ and $\langle \overline{12} \rangle$ are of all proper S-essential submodule of Z_{24} where $S = \{\overline{1}, \overline{3}, \overline{9}\}$.*

Proposition 3.1. *If $M = \oplus_{i \in I} N_i$ where N_i is a simple submodule of M such that $Ann_R(N_i) \cap S \neq \varnothing$ for all $i \in I$, then M does not contain proper S-essential submodule.*

Proof: Let E be an S-essential submodule of M and $i \in I$. Since $Ann_R(N_i) \cap S \neq \varnothing$, $N_i s \neq 0$ for all $s \in S$. However, E is an S-essential submodule of M, $E \cap N_i \neq 0$. Since N_i is a simple submodule of M, $E \cap N_i = N_i$ and thus $N_i \ll E$. So $M = \oplus_{i \in I} N_i \subseteq E$. Hence, $E = M$. Therefore, M does not contain proper S-essential submodule. □

Theorem 3.1. *Let $\{M_i\}_{i\in I}$ be a family of right R-modules indexed by a nonempty set I such that for each $m \in M - \{0\}$, $Ann_R(m) \cap S = \varnothing$ where $M = \oplus_{i\in I} M_i$. If there exists $i \in I$ such that M_i has a proper S-essential submodule, then M has a proper S-essential submodule.*

Proof: Suppose that there exists $i \in I$ such that M_i has a proper S-essential submodule E_i. Let $X = \oplus_{j\in I} X_j$ where $X_j = E_i$, if $j = i$ and $X_j = M_j$, if $j \neq i$. Let N be a submodule of M such that $Ns \neq 0$ for all $s \in S$. Let $(n_j)_{j\in I}$ be a non-zero element in N.

Case 1. $n_i = 0_{M_i}$. Then $0 \neq (n_j)_{j\in I} \in N \cap X$.

Case 2. $n_i \neq 0_{M_i}$. Since $Ann_R(m) \cap S = \varnothing$ for all $m \in M - \{0\}$, $n_i s \neq 0$ for all $s \in S$ and thus $n_i Rs \neq 0$ for all $s \in S$. However, X_i is an S-essential submodule of M_i, and we have $n_i R \cap X_i \neq 0$. So $X \cap N \neq 0$.

From Case 1 and Case 2, we have $X \cap N \neq 0$. Therefore, X is a proper S-essential submodule of M. □

Theorem 3.2. *Let $\{M_j\}_{j\in I}$ be a family of right R-modules indexed by a nonempty set I. If $M = \oplus_{j\in I} M_j$ has a proper S-essential submodule, then there exists $i \in I$ such that M_i has a proper S-essential submodule.*

Proof: Suppose that $M = \oplus_{j\in I} M_j$ has a proper S-essential submodule E and M_j does not have a proper S-essential submodule for each $j \in I$. Let $i \in I$ such that $E \cap M_i \neq M_i$ and L a submodule of M_i such that $E \cap M_i \cap L = 0$. So $E \cap L = 0$. Since E is an S-essential submodule of M, there exists $s \in S$ such that $Ls = 0$. We have $E \cap M_i$ is an S-essential submodule of M_i. This means that for each $i \in I$, $E \cap M_i$ is a proper S-essential submodule of M_i or $E \cap M_i = M_i$. Let $i \in I$. By assumption, we have $E \cap M_i = M_i$. So $M_i \subseteq E$ for all $i \in I$ and thus $M = E$. This is a contradiction. □

Let M be a right R-module and S a multiplicatively closed subset of a ring R such that $Ann_R(M) \cap S = \varnothing$. Set $\overline{S} = \{s + Ann_R(M) | s \in S\}$. However, $1 + Ann_R(M) \in \overline{S}$, $\overline{S} \neq \varnothing$. Let $s_1 + Ann_R(M), s_1 + Ann_R(M) \in \overline{S}$. Since S is a multiplicatively closed subset of a ring R, $(s_1 + Ann_R(M))(s_2 + Ann_R(M)) = s_1 s_2 + Ann_R(M) \in \overline{S}$. Since $Ann_R(M) \cap S = \varnothing$, $0 \notin \overline{S}$. Hence, $\overline{S}$ is a multiplicatively closed subset of a ring, $R/Ann_R(M)$.

Theorem 3.3. *Let M be a nonzero right R-module such that $Ann_R(m) \cap S = \varnothing$ for all $m \in M - \{0\}$. The following statements are equivalent.*

(i) *M has a proper S-essential submodule.*
(ii) *A right $R/Ann_R(M)$-module M has a proper $\overline{S}$-essential submodule where $\overline{S} = \{s + Ann_R(M) | s \in S\}$.*

Proof: (i) $\Longrightarrow$ (ii) Let E be a proper S-essential submodule of M and N a submodule of right $R/Ann_R(M)$-module M such that $Ns \neq 0$ for all $s \in \overline{S}$. Then $N \neq 0$. Choose $0 \neq n \in N$. Since $Ann_R(m) \cap S = \varnothing$ for all $m \in M - \{0\}$, $nRs \neq 0$ for all $s \in S$. However, E is an S-essential submodule of M, we have $nR \cap E \neq 0$. Choose $0 \neq nr \in nR \cap E$. Then $0 \neq n(r + Ann_R(M)) = nr \in nR \cap E \subseteq N \cap E$ and thus $N \cap E \neq 0$. Hence, E is a proper submodule of right $R/Ann_R(M)$ module M and E is an $\overline{S}$-essential submodule of M.

(ii) $\Longrightarrow$ (i) Let E be an $\overline{S}$-essential submodule of right $R/Ann_R(M)$ module M and N a submodule of M such that $Ns \neq 0$ for all $s \in S$. Then $N \neq 0$. Let $n \in N - \{0\}$. Since $Ann_R(m) \cap S = \varnothing$ for all $m \in M - \{0\}$, $ns \neq 0$ for all $s \in S$ and thus $n(s + Ann_R(M)) \neq 0$ for all $s \in S$. So $nR/Ann_R(M)(s + Ann_R(M)) \neq 0$ for all $s \in S$. However, E is an $\overline{S}$-essential submodule of right $R/Ann_R(M)$ module M, $nR/Ann_R(M) \cap E \neq 0$. There exists $r \in R$ such that $0 \neq n(r + Ann_R(M)) \in E$. So $nr \in E \cap N$ and then $E \cap N \neq 0$. Hence, E is an S-essential submodule of M. □

4. **Conclusion.** In this paper, we proved that the direct summand and intersection of S-essential submodules is also an S-essential submodule. Moreover, we showed that K

is an S-essential submodule of M if and only if the inclusion map $i_K : K \longrightarrow M$ is S-essential monomorphism if and only if for each right R-module N and $f \in Hom_R(M, N)$, if $(Ker(f)) \cap K = 0$ then there exists $s \in S$ such that $(Ker(f))s = 0$. In addition, the study revealed that monomorphism f from right R-module L to right R-module M, f is S-essential if and only if for each right R-module N and for each R-homomorphism $h : M \longrightarrow N$ such that $h \circ f$ is monomorphism there exists $s \in S$ such that $(Ker(h))s = 0$. Finally, for a nonzero right R-module M such that $Ann_R(m) \cap S = 0$ for all $m \in M - \{0\}$, we can show that M has a proper S-essential submodule if and only if a right $R/Ann_R(M)$-module M has a proper $\overline{S}$-essential submodule where $\overline{S} = \{s+Ann_R(M)|s \in S\}$.

In the future work, we would like to introduce and study the concept of S-uniform modules where S is a multiplicatively closed subset of a ring R. A right R-module M is called *S-uniform module*, if for any nonzero submodule of M is an S-essential submodule of M. The notion is natural generalizations of the classical notion of uniform module.

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