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OBSERVER-BASED FINITE-TIME ADAPTIVE CONTROL FOR MIMO NONLINEAR QUANTIZED SYSTEMS

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Abstract. *This paper investigates the adaptive fuzzy control for a class of multi-input and multi-output (MIMO) nonlinear systems which inputs are under the quantization action. The quantizer is assumed to be of the hysteresis type. Fuzzy logic systems (FLSs) are employed to identify the unknown nonlinearities. After constructing observers to estimate the unavailable states, a backstepping control scheme is developed. By usage of the finite-time Lyapunov stability theory, a fuzzy output-feedback controller is presented, which guarantees the semi-global practical finite-time stability (SGPFS) of all the signals in the closed-loop systems. And at the same time, the design controller also enables the tracking error converges to a small neighborhood of the origin in finite time.*
Keywords: Adaptive control, Backstepping technique, Fuzzy logical systems, Finite-time stability, Output-feedback control

1. **Introduction.** Since Peter Dorator put forward the concept of short-term stability in 1961 [1], finite-time control strategy has become a research hotspot. A great number of works have been widely carried out [1, 2, 3, 4, 5, 6]. However, these studies assume system states are known. These control methods are no longer applicable when system states are unavailable. Then the output-feedback control is worth considering. However, some existing observer design methods for Lipschitz continuous systems are not suitable for finite-time systems. Many scholars continue to study and seek different solutions. In [7], a finite-time observer for double integral systems by using homogeneous properties is designed, which is difficult to be extended to higher-order systems. In [8], a finite-time output-feedback control is discussed for a class of norm second-order nonlinear systems by using the power addition integral method, but the designed observer is complex. In [9], the observer-based adaptive control strategy is formed without the linear growth conditions for nonlinear terms. In [10, 11], the fuzzy state observer is constructed for MIMO nonlinear systems. The authors of [12] propose the finite-time prescribed performance control method, which not only solved the unmeasured states case but also simultaneously settled the issue of the explosion of complexity.

Motivated by the above observations, this note concentrates on the finite-time output-feedback controller design for a class of MIMO nonlinear systems with the quantization action. By applying the finite-time Lyapunov stability theorem and the backstepping

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method, state observers are constructed and the output-feedback control scheme is discussed to ensure the performance of the system in finite time. By combining with convex combination method, the stability analysis depends on the solvability of a set of linear matrix inequalities, which can greatly reduce the computational complexity. In contrast with the existing finite-time control methods, the current work is more adaptable to the realistic systems.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given. Finite-time control design is completed in Section 3. A simulation example is discussed in Section 4. At last, we conclude this work.

2. **Problem Statement and Preliminaries.** The following MIMO nonlinear systems with the ith ($i = 1, 2, \ldots, N$) subsystem are considered:

$$\begin{cases} \dot{x}_{i,j} = f_{i,j}\left(\bar{x}_{i,j}\right) + x_{i,j+1} + d_{i,j}(x), \quad 1 \le j \le n_i - 1, \\ \dot{x}_{i,n_i} = f_{i,n_i}(x) + q(u_i) + d_{i,n_i}(x), \\ y_i = x_{i,1} \end{cases} \tag{1}$$

where $\bar{x}_{i,j} = [x_{i,1}, x_{i,2}, \ldots, x_{i,j}]^T \in R^j$, $(i = 1, \ldots, N;\ j = 1, \ldots, n_i)$. $d_{i,j}(\cdot)$ is the external disturbance satisfying $d_{i,j}(\cdot) \le \bar{d}_{i,j}$ with $\bar{d}_{i,j}$ being positive constant. $f_{i,j}\left(\bar{x}_{i,j}\right)$ is continuously differentiable with $f_{i,j}(\cdot) = 0$. The output $y_i \in R$ can be measured directly only. $u_i \in R$ is the controller to be designed to the ith subsystem which is supposed to be limited by a large positive constant M_i. The system input $q(u_i)$ under the quantization action is defined as follows:

$$q(u_i) = \begin{cases} \mu_{m_i} sgn(u), & \frac{\mu_{m_i}}{1+\delta_q} < |u_i| \le \mu_{m_i}, \dot{u}_i < 0, \text{ or } \mu_{m_i} < |u_i| \le \frac{\mu_{m_i}}{1-\delta_q}, \dot{u}_i > 0, \\ \mu_{m_i}\left(1+\delta_q\right) sgn(u_i), & \mu_{m_i} < |u_i| \le \frac{\mu_{m_i}}{1-\delta_q}, \dot{u}_i < 0, \text{ or } \frac{\mu_{m_i}}{1-\delta_q} < |u_i| \le \frac{\mu_{m_i}(1+\delta_q)}{1-\delta_q}, \\ 0, & 0 \le |u_i| < \frac{\mu_{\min}}{1+\delta_q}, \dot{u}_i < 0, \text{ or } \frac{\mu_{\min}}{1+\delta_q} \le |u_i| \le \mu_{\min}, \dot{u}_i > 0, \\ q(u_i(t^-)), & \dot{u}_i = 0 \end{cases}$$

where $\mu_{m_i} = \rho^{1-i}\mu_{\min}$, $m_i = 1, 2, \ldots$, $\delta_q = \frac{1-\rho}{1+\rho}$. $\mu_{\min} > 0$ determines the dead-zone range of $q(u_i)$. $0 < \rho < 1$ denotes the quantization density, $q(u_i)$ is in the set $U = \{0, \pm\mu_{m_i}, \pm\mu_{m_i}\left(1+\delta_q\right)\}$.

Assumption 2.1. *For $1 \le i, m \le N$, $1 \le j \le n_i$, $1 \le n \le n_m$, there exist known constants $\underline{a}_{pq}$ and $\overline{a}_{pq}$ such that $\underline{a}_{pq} \le \frac{\partial f_{i,j}}{\partial x_{m,n}} \le \overline{a}_{pq}$. n_i and n_m stand for the number of state variables in the ith and jth subsystems, respectively. $p = \sum_{k=0}^{i-1} n_k + j$ and $q = \sum_{k=0}^{m-1} n_k + n$ with $n_0 = 0$.*

Remark 2.1. *Since $f_{i,j}(x) = \left[\frac{\partial f_{i,j}}{\partial x_{1,1}}, \ldots, \frac{\partial f_{i,j}}{\partial x_{N,n_N}}\right] x$. By Assumption 2.1, there exists constant $h_{i,j} > 0$ such that $|f_{i,j}(x)| \le h_{i,j}||x||$. Then $\Phi_{i,j}(w) = h_{i,j}w$, with $w \in R$, is the bounding function of $f_{i,j}(\cdot)$.*

Assumption 2.2. *The desired trajectories of the ith subsystem y_{di} and its k-order derivative $y_{di}^{(k)}$ are continuous and bounded. That is, $|y_{di}| < \bar{y}_{di}$ and $\left|y_{di}^{(k)}\right| < \bar{y}_{di}$, with $\bar{y}_{di}$ being a positive constant.*

The quantized input $q(u_i)$ can be represented as $q(u_i) = H(u_i)u_i(t) + L(t)$, where $1 - \delta_q \le H(u_i) \le 1 + \delta_q$, $|L(t)| \le \mu_{\min}$. With $\rho(t) \le \frac{1+\delta_q}{1-\delta_q} - 1$, one has $q(u_i) = \left(1 - \delta_q\right)(1 + \rho(t))u_i(t) + L(t)$. Furthermore, the systems (1) can be rewritten as the following form (for $1 \le i \le N$)

$$\begin{cases} \dot{x}_i = A_i x_i + L_i y_i + F_i(x) + d_i + B_i\left(1 - \delta_q\right)u_i(t) + B_i\left[\left(1 - \delta_q\right)\rho(t)u_i(t) + L(t)\right], \\ y_i = C_i^T x_i \end{cases} \tag{2}$$

where $F_i(x) = [f_{i,1}(x_{i,1}), \ldots, f_{i,n_i}(x)]^T$, $L_i = [l_{i,1}, \ldots l_{i,n_i}]^T$, $B_i = [0, \ldots, 0, 1]^T_{n_i \times 1}$, $C_i^T = [1, 0, \ldots, 0]_{1\times n_i}$, $d_i = [d_{i,1}, \ldots, d_{i,n_i}]^T$, and $A_i = \begin{bmatrix} L_{n_i-1} & I_{n_i-1} \\ -l_{i,n_i} & 0 \end{bmatrix}$ with $L_{n_i-1} = [-l_{i,1}, \ldots, -l_{i,n_i-1}]^T$. The vector L_i is chosen suitably such that A_i is a strict Hurwitz matrix.

3. **Finite-Time Fuzzy Control Design and Stability Analysis.** In this section, the state observer is constructed firstly. Let $\hat{x}_{i,j}$ denote the estimation of $x_{i,j}$ and $e_{i,j} = x_{i,j} - \hat{x}_{i,j}$ denotes the estimation error. The state observer for the ith subsystem can be designed as

$$\begin{cases} \dot{\hat{x}}_{i,j} = \hat{x}_{i,j+1} + l_{i,j}e_{i,1}, \quad 1 \le i \le N,\ 1 \le j \le n_i - 1, \\ \dot{\hat{x}}_{i,n_i} = u_i + l_{i,n_i}e_{i,1} \end{cases} \tag{3}$$

The state observer (3) can be rewritten as $\dot{\hat{x}}_i = A_i\hat{x}_i + L_iy_i + B_iu_i$. Furthermore, the observer error equation can be got: $\dot{e}_i = A_ie_i + F_i(x) + d_i + B_i\overline{u_i} + B\underline{u_i}$, with $e_i = [e_{i,1}, \ldots, e_{i,n_i}]^T$. Let $\overline{u_i} = (1-\delta_q)\,u_i(t) - u_i(t)$, $\underline{u_i} = (1-\delta_q)\,\rho(t)u_i(t) + L(t)$, $e = \left[e_1^T, \ldots, e_N^T\right]^T$, $A = diag[A_1, \ldots, A_N]$, $F(x) = \left[F_1^T(x), \ldots, F_N^T(x)\right]^T$, $B = diag[B_1, \ldots, B_N]$, $D = \left[d_1^T, \ldots, d_N^T\right]^T$. The whole observer-error equation can be expressed by $\dot{e} = Ae + F(x) + D + B(\bar{u} + \underline{u})$.

Next up, the coordinate transformation $z_{i,j} = \hat{\chi}_{i,j} - \alpha_{i,j-1}$ is needed, where $\alpha_{i,0} = y_{di}$. $\alpha_{i,j}$ $(1 \le j \le n_i)$ is the control signal. Consider the following Lyapunov function candidate $V = V_e + \sum_{i=1}^N V_z + \sum_{i=1}^N V_\theta$ with $V_e = e^TPe$, $V_z = \frac{1}{2}\sum_{j=1}^{n_i} z_{i,j}^2$, $V_\theta = \frac{1}{2r_i}\tilde{\theta}_i^2$. $\tilde{\theta}_i$ will be defined later. First, the time derivative of V_z is calculated as

$$\begin{aligned} \dot{V}_z = {} & z_{i,1}\left(z_{i,2} + \alpha_{i,1} + l_{i,1}e_{i,1} - \dot{y}_{di}\right) + \sum_{j=2}^{n_i-1} z_{i,j}\left(\alpha_{i,j} - \sum_{k=1}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial\hat{x}_{i,k}}\hat{x}_{i,k+1}\right. \\ & \left. - \frac{\partial\alpha_{i,j-1}}{\partial\hat{\theta}_i}\dot{\hat{\theta}}_i - \sum_{k=0}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial {y_{di}}^{(k)}}y_{di}^{(k+1)}\right) + \sum_{j=2}^{n_i-1} z_{i,j}e_{i,1}\left(l_{i,j} - \sum_{k=1}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial\hat{x}_{i,k}}l_{i,k}\right) \\ & + \sum_{j=2}^{n_i-1} z_{i,j}z_{i,j+1} + z_{i,n_i}\left(v_i - \dot{\alpha}_{n_i-1} + l_{i,n_i}e_{i,1}\right) \end{aligned} \tag{4}$$

By using the completion of squares, one has $z_{i,1}l_{i,1}e_{i,1} \le \frac{1}{2\beta_{i,1}}l_{i,1}^2z_{i,1}^2 + \frac{\beta_{i,1}}{2}e_{i,1}^2$ and $z_{i,j}e_{i,1}\Big(l_{i,j} - \sum_{k=1}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial\hat{x}_{i,k}}l_{i,k}\Big) \le \frac{1}{2\beta_{i,j}}z_{i,j}^2\left(l_{i,j} - \sum_{k=1}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial\hat{x}_{i,k}}l_{i,k}\right)^2 + \frac{1}{2}\beta_{i,j}e_{i,1}^2$. Moreover, let $\bar{f}_{i,1}(Z_{i,1}) = \frac{1}{2\beta_{i,1}}z_{i,1}l_{i,1}^2 + cz_{i,1}\phi_{i,1}^2\left(\hat{\theta}_{i,1}\right) - \dot{y}_{di} + \frac{1}{2}z_{i,1} + c\psi_{i,1}z_{i,1}^{2(2\gamma-1)-1}$, $\bar{f}_{i,j}(Z_{i,j}) = -\sum_{k=1}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial\hat{x}_{i,k}}\hat{x}_{i,k+1} - \frac{\partial\alpha_{i,j-1}}{\partial\hat{\theta}_i}\dot{\hat{\theta}}_i + \frac{1}{2\beta_{i,j}}z_{i,j}\left(\sum_{k=1}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial\hat{x}_{i,k}}l_{i,k} - l_{i,j}\right)^2 + \frac{1}{2}z_{i,j} + c\psi_{i,j}z_{i,j}^{2(2\gamma-1)-1} - \sum_{k=0}^{j-1}\frac{\partial\alpha_{i,j-1}}{\partial {y_{di}}^{(k)}}y_{di}^{(k+1)} + z_{i,j-1} + cz_{i,j}\phi_{i,j}^2\left(\hat{\theta}_{i,j}\right)$, and $\bar{f}_{i,n_i}(Z_{i,n_i}) = -\dot{\alpha}_{i,n_i-1} + l_{i,n_i}e_{i,1} + z_{i,n_i-1} + cz_{i,n_i}\phi_{i,n_i}^2\left(\hat{\theta}_{i,n_i}\right) + \frac{1}{2}z_{i,n_i} + c\psi_{i,n_i}z_{i,n_i}^{2(2\gamma-1)-1}$. $\gamma = \frac{2\varsigma-1}{2\varsigma+1}$, with $\varsigma \ge 2$ being a positive integer. c, $\phi_{i,j}\left(\hat{\theta}_{i,j}\right)$ and $\psi_{i,j}$ will be defined later. Now, considering these $\bar{f}_{i,j}(\cdot)s$ and substituting the above two inequalities into (4), one has

$$\begin{aligned} \dot{V}_z \le {} & \sum_{j=1}^{n_i} z_{i,j}\left(\alpha_{i,j} + \bar{f}_{i,j}(z_{i,j})\right) + \sum_{j=1}^{n_i-1}\frac{1}{2\beta_{i,j}}e_{i,1}^2 - c\sum_{j=1}^{n_i} z_{i,j}^2\phi_{i,j}^2\left(\hat{\theta}_{i,j}\right) \\ & - \sum_{j=1}^{n_i}\frac{1}{2}z_{i,j}^2 - c\sum_{j=1}^{n_i}\psi_{i,j}z_{i,j}^{2(2\gamma-1)} \end{aligned} \tag{5}$$

The unknown functions $\bar{f}_{i,j}(z_{i,j})s$ can be modeled by the FLSs $W_{i,j}^T S_{i,j}(Z_{i,j})$ based on Lemma 4 in [13]. For any given $\varepsilon_{i,j} > 0$, such that $\bar{f}_{i,j}(Z_{i,j}) = W_{i,j}^T S_{i,j}(Z_{i,j}) + \delta_{i,j}(Z_{i,j})$. $\delta_{i,j} \leq \varepsilon_{i,j}$ denotes the approximation error. $S_{i,j}(Z_{i,j})$ is the basis function vector. Using completion of squares again, one has

$$z_{i,j}\bar{f}_{i,j} \leq \frac{1}{2a_{i,j}^2} z_{i,j}^2 \theta_i S_{i,j}^T(Z_{i,j}) S_{i,j}(Z_{i,j}) + \frac{1}{2}a_{i,j}^2 + \frac{1}{2}z_{i,j}^2 + \frac{1}{2}\varepsilon_{i,j}^2, \quad 1 \leq i \leq N,\ 1 \leq j \leq n_i \tag{6}$$

where the unknown constant $\theta_i = \max_{1\leq j\leq n_i}\{\theta_{i,j}\}$ with $\theta_{i,j} = \|W_{i,j}\|^2$. Construct the finite-time fuzzy control signals as follows (for $i = 1, \ldots, N$; $j = 1, \ldots, n_i$):

$$\alpha_{i,j} = -\frac{1}{2a_{i,j}^2} z_{i,j}\hat{\theta}_i S_{i,j}(Z_{i,j})^T S_{i,j}(Z_{i,j}) - k_{i,j} z_{i,j}^{2\gamma-1} \tag{7}$$

where $k_{i,j}$ and $a_{i,j}$ are positive design parameters. $\hat{\theta}_i$ is the estimation of θ_i. The evaluated error is $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$. For any initial condition $\hat{\theta}_i(t_0) \geq 0$, the solution $\hat{\theta}_i(t) \geq 0$ holds for $t \geq t_0$. In the following text, it is assumed that $\hat{\theta}_i(t) \geq 0$. Then substituting (7) and (6) into (5), one has

$$\begin{aligned} \dot{V}_z \leq & -\sum_{j=1}^{n_i} k_{i,j} z_{i,j}^{2\gamma} + \sum_{j=1}^{n_i} \frac{1}{2a_{i,j}^2} z_{i,j}^2 \tilde{\theta}_i S_{i,j}^T(Z_{i,j}) S_{i,j}(Z_{i,j}) + \sum_{j=1}^{n_i} \frac{1}{2}\left(a_{i,j}^2 + \varepsilon_{i,j}^2\right) \\ & + \sum_{j=1}^{n_i-1} \frac{1}{2}\beta_{i,j} e_{i,1}^2 - c\sum_{j=1}^{n_i} z_{i,j}^2 \phi_{i,j}^2\left(\hat{\theta}_{i,j}\right) - c\sum_{j=1}^{n_i} \psi_{i,j} z_{i,j}^{2(2\gamma-1)-1} \end{aligned} \tag{8}$$

Next up, calculate the derivative of $V_e = e^T P e$.

$$\begin{aligned} \dot{V}_e = & e^T\left(PA + A^T P\right)e + 2e^T P(F(x) - F(\hat{x})) + 2e^T P F(\hat{x}) + 2e^T P D \\ & + 2e^T P B\left(\bar{u} + \underline{u}\right) \end{aligned} \tag{9}$$

With the fact $P > 0$, we have $2e^T P(F(x) - F(\hat{x})) = 2e^T P J e \leq e^T\left[PJ + J^T P\right]e$, where $J = \left[\frac{\partial f_{i,j}}{\partial x_{m,n}}\right]$ is a Jacobian matrix, which has g rows and g columns. According to Assumption 2.2, every nonzero element in the matrix has its own upper and lower bounds. Namely, there exists a function $0 \leq \mu_{pq}(t) \leq 1$ such that $\frac{\partial f_{i,j}}{\partial x_{m,n}} = \mu_{pq}\underline{a}_{pq} + (1-\mu_{pq})\bar{a}_{pq}$. Thus, J can be reformulated as $J = \sum_{p=1}^{g}\sum_{q=1}^{g}\left[\mu_{pq}\underline{F}_{pq} + (1-\mu_{pq})\overline{F}_{pq}\right]$, $0 < \alpha_{pq} < 1$, where $\underline{F}_{pq}$ and $\overline{F}_{pq}$ are constant matrixes and they have only one nonzero element $\underline{a}_{pq}$ and $\bar{a}_{pq}$ at their pth row and qth column, respectively.

Next up, from Remark 2.1 and Lemma 3 in [14], one has $2e^T PF(\hat{x}) \leq \varepsilon_0 e^T e + c\left(\sum_{i=1}^{N}\sum_{j=1}^{n_i}|z_{i,j}|^2\phi_{i,j}^2\left(\hat{\theta}_{i,j}\right)\right) + c\left(\sum_{i=1}^{N}\sum_{j=1}^{n_i}|z_{i,j}|^{2(2\gamma-1)}\psi_{i,j}^2\right) + \sum_{i=1}^{N} c_0\bar{y}_{di}$, with $c_0 = \varepsilon_0^{-1}\|P^2\|\sum_{j=1}^{n_i-1}h_{i,j}^2$ and $c = gc_0$. $\phi_i\left(\hat{\theta}_i\right) = \frac{1}{2a_i^2}|z_i|\hat{\theta}_i s^2 + 1$ and $\psi_i = k_i$; for $i = n$, $\phi_i\left(\hat{\theta}_i\right) = 1$ and $\psi_i = 1$. Furthermore, for any positive constant τ, based on Assumption 2.1, one has $2e^T PD \leq \tau e^T e + \frac{1}{\tau}\|P\|^2\sum_{i=1}^{N}\sum_{j=1}^{n_i}\bar{d}_{ij}^2$, $2e^T PB\bar{u} \leq \tau e^T e + \frac{1}{\tau}\|P\|^2\sum_{i=1}^{N}\delta_q^2 M_i^2$, $2e^T PB\underline{u} \leq \tau e^T e + \frac{1}{\tau}\|P\|^2\sum_{i=1}^{N}(2\delta_q M_i + \mu_{\min})^2$, $2e^T PB\underline{u} \leq \tau e^T e + \frac{1}{\tau}\|P\|^2\sum_{i=1}^{N}(2\delta_q M_i + \mu_{\min})^2$. Let $\delta_0 = \frac{1}{\tau}\|P\|^2\sum_{i=1}^{N}\sum_{j=1}^{n_i}\bar{d}_{ij}^2 + \frac{1}{\tau}\|P\|^2\sum_{i=1}^{N}\left(\underline{\beta}_i - 1\right)^2 W_i^2 + \frac{1}{\tau}\|P\|^2\sum_{i=1}^{N}\delta_q^2 M_i^2 + \sum_{i=1}^{N}c_0\bar{y}_{di}$ and $\bar{P} = PA + A^T P + PJ + J^T P + (\varepsilon_0 + 3\tau)I$. Substituting the above four inequalities into (9) we have

$$\dot{V}_e \leq e^T\bar{P}e + \delta_0 + c\left(\sum_{i=1}^{N}\sum_{j=1}^{n_i}|z_{i,j}|^{2(2\gamma-1)}\psi_{i,j}^2\right) + c\left(\sum_{i=1}^{N}\sum_{j=1}^{n_i}z_{i,j}^2\phi_{i,j}^2\left(\hat{\theta}_{i,j}\right)\right) \tag{10}$$

Notice $\dot{V}_\theta = -\frac{1}{r_i}\tilde{\theta}_i\dot{\hat{\theta}}_i$, and let $\hat{\theta}$ satisfy the following differential equations:

$$\dot{\hat{\theta}}_i = \sum_{j=1}^{n_i} \frac{r_i}{2a_{i,j}^2} z_{i,j}^2 S_{i,j}(Z_{i,j})^T S_{i,j}(Z_{i,j}) - \sigma_i \hat{\theta}_i \tag{11}$$

where r_i and σ_i are positive design parameters. Taking (10) and (8) into account, the derivative of the whole Lyapunov candidate is as follows.

$$\dot{V} \le e^T \left(\bar{P} + \beta\right) e - \sum_{i=1}^{N} \sum_{j=1}^{n_i} k_{i,j} z_{i,j}^{2\gamma} + \bar{\delta}_0 + \sum_{i=1}^{N} \frac{\sigma_i}{r_i} \tilde{\theta}_i \hat{\theta}_i \tag{12}$$

where $\beta = diag\left[\sum_{i=1}^{N} \sum_{j=1}^{n_i-1} \frac{1}{2}\gamma_{i,j}, 0, \ldots, 0\right]$. $\bar{\delta}_0 = \bar{\delta} + \sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{1}{2}\left(a_{i,j}^2 + \varepsilon_{i,j}^2\right)$.

At this stage, the main results are summarized in the following theorem.

Theorem 3.1. *Based on Assumptions 2.1 and 2.2 and the following inequality (13), the control signals $\alpha_{i,j}$ (7), state observer systems (3) and adaptive laws θ_i (11), are designed for considering nonlinear MIMO system (1). All the signals of the close-loop nonlinear MIMO system are SGPFS, and tracking errors can converge to a small neighbourhood of the origin in finite time.*

$$PA + A^T P + PJ_{pq} + J_{pq}^T P + \varepsilon_0 I + 3\tau I + \beta < 0, \quad 1 \le p, q \le g \tag{13}$$

where $g = \sum_{i=1}^{N} n_i$ and P is a definitive positive matrix. J_{pq} is a constant matrix which element at the pth row and the qth column is $\bar{a}_{pq}$ or $\underline{a}_{pq}$ and others are zero.

Proof: According to Lemma 3 in [15], (13) is equivalent to the following inequity $\bar{P} + \beta < 0$. It means that there exists a constant $\mu > 0$, such that $e^T\left(\bar{P} + \beta\right)e < -\frac{\mu}{\lambda_M(P)} e^T Pe$, where $\lambda_M(P)$ is the maximal eigenvalue of matrix P. Next, using $\tilde{\theta}\hat{\theta} \le -\frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}\theta^2$, we can obtain

$$\dot{V} \le -\frac{\mu}{\lambda_M(P)} e^T Pe - \sum_{i=1}^{N} \sum_{j=1}^{n_i} k_{\min} z_{i,j}^{2\gamma} - \sum_{i=1}^{N} \frac{\sigma_{\min}}{2r_i} \tilde{\theta}_i^2 + \sum_{i=1}^{N} \frac{\sigma_i}{2r_i} \theta_i^2 + \bar{\delta}_0 \tag{14}$$

where $k_{\min} = \sum_{1 \le i \le N}^{1 \le j \le n_i} \{k_{i,j}\}$, $\sigma_{\min} = \sum_{1 \le i \le n} \{\sigma_i\}$. By Lemma 3 in [13], let $m = 1 - \gamma$, $n = \gamma$, $w = e^{\frac{\gamma}{1-\gamma}\ln\gamma}$, $\xi = 1$, $\kappa = \sum_{j=1}^{n_i} \frac{1}{2r_i}\tilde{\theta}_i^2$, we have $\left(e^T Pe\right)^\gamma \le (1-\gamma)e^{\frac{\gamma}{1-\gamma}\ln\gamma} + e^T Pe$ and $-\frac{\alpha}{\lambda_M(P)} e^T Pe \le -\frac{\alpha}{\lambda_M(P)}\left(e^T Pe\right)^\gamma + \frac{\alpha}{\lambda_M(P)}(1-\gamma)e^{\frac{\gamma}{1-\gamma}\ln\gamma}$. In the same way, we have $\left(\sum_{j=1}^{n_i} \frac{1}{2r_i}\tilde{\theta}_i^2\right)^\gamma \le (1-\gamma)e^{\frac{\gamma}{1-\gamma}\ln\gamma} + \sum_{j=1}^{n_i} \frac{1}{2r_i}\tilde{\theta}_i^2$ and $-\sigma_{\min}\sum_{j=1}^{n_i} \frac{1}{2r_i}\tilde{\theta}_i^2 \le -\sigma_{\min}\left(\sum_{j=1}^{n_i} \frac{1}{2r_i}\tilde{\theta}_i^2\right)^\gamma$ $+ \sigma_{\min}(1-\gamma)e^{\frac{\gamma}{1-\gamma}\ln\gamma}$. Moreover, let $\bar{k} = 2^\gamma k_{\min}$ and using Lemma 2 in [13], we have $-k_{\min}\sum_{j=1}^{n_i} z_{i,j}^{2\gamma} \le -\bar{k}\left(\frac{1}{2}\sum_{i=j}^{n_i} z_{i,j}^2\right)^\gamma$. Define $\lambda = \min\left\{\bar{k}, \frac{\alpha}{\lambda_M(P)}, \sigma_{\min}, i = 1, 2, \ldots, n\right\}$, $\eta = \left(\frac{\alpha}{\lambda_M(P)} + \sigma_{\min}\right)(1-\gamma)e^{\frac{\gamma}{1-\gamma}\ln\gamma} + \sum_{i=1}^{n} \frac{b_i\sigma_i}{2r_i}\theta_i^2 + \bar{\delta}_0$. Submitting the above three inequalities into (14), one has

$$\dot{V}_n \le -\lambda V^\gamma + \eta \tag{15}$$

Let $T_{reach} = \frac{1}{(1-\gamma)\theta_0\lambda}\left[V^{1-\gamma}\left(z(0), \tilde{\theta}(0)\right) - \left(\frac{\eta}{(1-\theta_0)\lambda}\right)^{\frac{1-\gamma}{\gamma}}\right]$ and by Lemma 1 in [13], for $t \ge T_{reach}$, we have $V^\gamma\left(z(t), \bar{\theta}(t)\right) \le \frac{\eta}{(1-\theta_0)\lambda}$. That is to say, all the signals in the systems are SGPFS. Moreover, for $t \ge T_{reach}$, we can obtain $|y - y_{dr}| \le 2\left(\frac{\eta}{(1-\theta_0)\lambda}\right)^{\frac{1}{2\gamma}}$, which means the tracking error converges to a small neighborhood of the origin after a finite time T_{reach}. The proof is finished.

4. **Simulation Example.** Consider the following MIMO systems:

$$\begin{cases} \dot{x}_{i,1} = x_{i,2} + f_{i,1}(x_{i,1}) + d_{i,1}(x), \\ \dot{x}_{i,2} = 0.6q(u_i) + f_{i,2}(x) + d_{i,2}(x), \\ y_i = x_{i,1} \end{cases} \quad (16)$$

where $i = 1, 2$, $f_{1,1} = 0.5\cos(x_{11})\cos(0.5x_{11})$, $f_{1,2} = -2.5\sin(x_{1,2})\sin(x_{1,2})$, $f_{2,1} = 0.5\sin^2(x_{21})$, $f_{2,2} = -1.5\sin(x_{2,2})\cos(x_{2,2})$, and $d_{i,1} = 0.22\sin(t)$, $d_{i,2} = -0.05\sin(t)$. where $q(u_i)$ is the quantized input with $\delta_q = 0.3$ and $\mu_{\min} = 0.1$. The reference signals are selected as $y_{d1} = \sin(0.5t) + 0.5\sin(t)$ and $y_{d2} = \sin(0.5t) + 0.5\sin(1.5t)$.

It is noticed that, the finite-time control method proposed in [6] cannot be utilized to control above system, because the system states $x_{i,2}$ are unmeasured. The observer-based finite-time control strategies in [12] cannot guarantee the system performance of (16), because the inputs of the system (16) are quantized. In our simulation, we apply seven fuzzy sets over the interval $[-1.5, 1.5]$ for each state variable. The initial conditions are chosen as $x_{i,1}(0) = \hat{x}_{i,1}(0) = 1$, $x_{i,2}(0) = \hat{x}_{i,2}(0) = -0.5$, $\hat{\theta}_{i,j}(0) = 0.2$. The design parameters are chosen as $\varepsilon_0 + 3\tau = 0.01$, $\beta = 0.01I$, $\gamma = 20/21$, $k_{i,j} = 10$, $a_{i,1} = 0.8$, $a_{i,2} = 1$, $\sigma_{i,1} = 1$, $\sigma_{i,2} = 2$, $r_{i,1} = 20$, $r_{i,2} = 25$. For the given constant matrix $J_{p,q}$, by solving LMIs, we can get $l_{1,1} = 22.8$, $l_{1,2} = 138.8$, $l_{2,1} = 15.2$, $l_{2,2} = 48.9$ and the positive definite matrix $P = diag[P_1, P_2]$ as

$$P_1 = \begin{bmatrix} 2.6369 & -0.4104 \\ -0.4104 & 0.0823 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2.4963 & -0.7205 \\ -0.7205 & 0.2601 \end{bmatrix}$$

Figures 1 and 3 reveal the system outputs follow the desired reference signals in a bounded set in finite time. And all the figures show the closed-loop signals are SGPFS. So the proposed fuzzy finite-time adaptive controllers are effective.

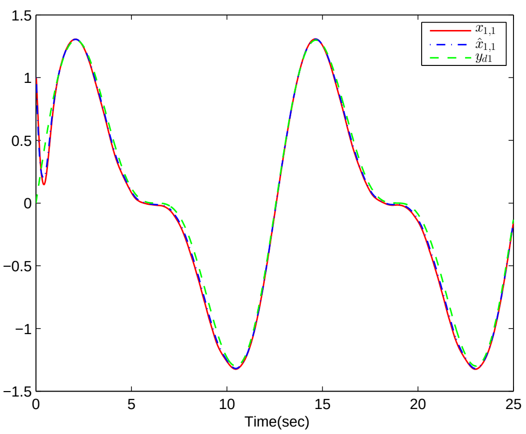

FIGURE 1. $x_{1,1}$, y_{d1}, $\hat{x}_{1,1}$

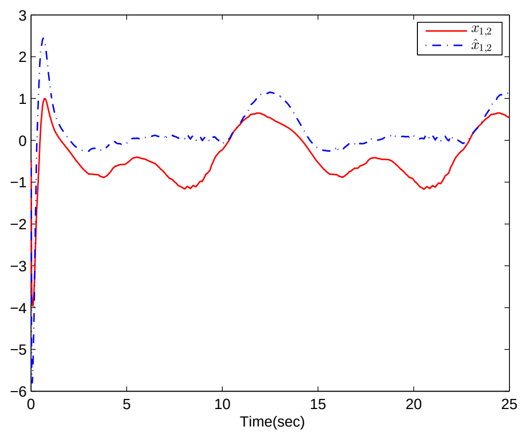

FIGURE 2. $x_{1,2}$, $\hat{x}_{1,2}$

5. **Conclusion.** In this note, the finite-time control problem for a class of nonlinear systems with input quantization is discussed. By usage of backstepping technique and the universal approximation of FLSs, an adaptive practical finite-time control strategy based on observer is proposed. The limitation of linear growth condition for nonlinear terms in existing results is further relaxed. Under the proposed control scheme, the tracking errors can converge to a small neighborhood of origin in finite time, which is verified by the example in simulation part. It is worth noting that the system studied in this paper is genial. It is more interesting how to extend the finite-time control method proposed in this paper to the practical engineering systems, which will be one of our future research directions.

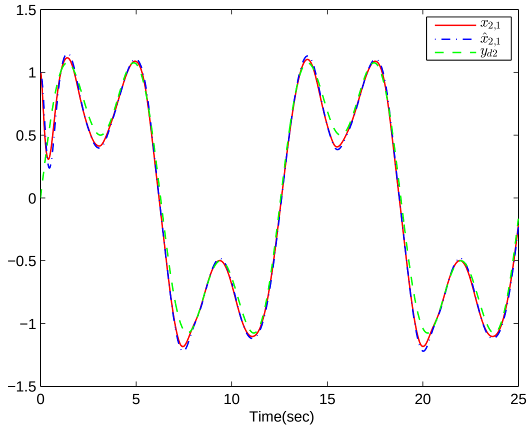

FIGURE 3. $x_{2,1}$, y_{d2}, $\hat{x}_{2,1}$

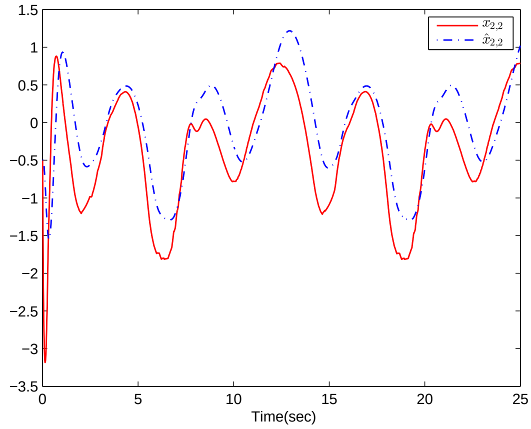

FIGURE 4. $x_{2,2}$, $\hat{x}_{2,2}$

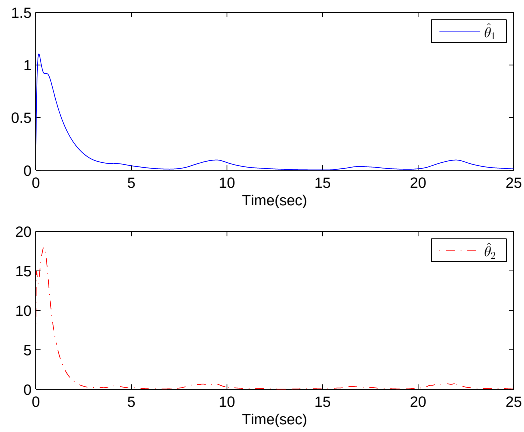

FIGURE 5. $\hat{\theta}_i$ $(i = 1, 2)$

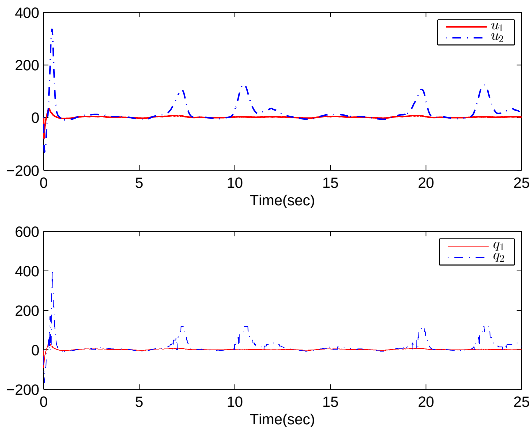

FIGURE 6. u_i, q_i

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