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HESITANT FUZZY BI-QUASI IDEALS IN SEMIGROUPS

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Abstract. *The concepts of bi-quasi ideals of semigroups were presented by Krishna Rao in 2017, and it is a generalization of left ideals, right ideals, bi-ideals, and quasi-ideals of semigroups. In this paper, we introduce the notion of hesitant fuzzy bi-quasi ideals of semigroups. We investigate the properties of hesitant fuzzy bi-quasi ideals of semigroups. In addition, we characterize the regular semigroups as hesitant fuzzy bi-quasi ideals of semigroups.*
Keywords: Regular semigroup, Hesitant fuzzy set, Hesitant fuzzy ideals, Hesitant fuzzy bi-quasi ideals

1. **Introduction.** In 1965, the fundamental concept of theory fuzzy sets was introduced by Zadeh [1]. It is an essential tool in science, engineering, computer science, control engineering, etc. In 1981, Kuroki [2, 3, 4] gave the idea of fuzzy ideals, fuzzy bi-ideals, and fuzzy interior ideals in semigroups. Later, concepts were expanded about interval valued fuzzy sets that have many applications such as approximate reasoning, image processing, decision making, medicine, and mobile networks. In 2006, Narayanan and Manikanran [5] initiated the notion of interval valued fuzzy ideals in semigroups. In 2010, Torra [6] proposed an extension of the notion so-called a hesitant fuzzy set which is a function from a reference set to a power set of the unit interval. Hesitant fuzzy set theory has been applied to several practical problems, primarily in the area of decision making. In 2015, Jun et al. [7] developed the notion of hesitant fuzzy sets to a semigroup. They introduced and applied the idea of hesitant fuzzy semigroups and hesitant fuzzy left (right) ideals, and investigated their properties. They considered characterizations of hesitant fuzzy subsemigroups and hesitant fuzzy left (right) ideals. The hesitant sets research has included many studies; for example, in 2020, Yiarayong [8] studied hesitant fuzzy sets in ternary semigroups. In 2021, Lekkoksung [9] discussed hesitant ideals in ordered semigroups. In 2022, Julatha and Iampan [10] gave the concepts of inf-hesitant fuzzy Γ-ideals in Γ-semigroups. In the same year, Kona and Rao [11] studided hesitant fuzzy right bi-quasi ideals of Γ-semigroups. In 2023, Kou et al. [12] studied k-hyperideals in ordred semihypering. The research of bi-quasi ideals and fuzzy bi-quasi ideals in semigroups studied in 2019 by Rao [13]. The aim of this paper is that we define the definition of the hesitant fuzzy bi-quasi ideals of semigroups, and we prove the properties of hesitant

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fuzzy bi-quasi ideals of semigroups. Furthermore, we characterize the regular semigroups in terms of a hesitant fuzzy bi-quasi ideal of a semigroup.

2. **Preliminaries.** In this section, we give definitions which are used in this paper.

By a subsemigroup of a semigroup S we mean a non-empty subset A of S such that $A^2 \subseteq A$, and by a left (right) ideal of S we mean a non-empty subset A of S such that $SA \subseteq A(AS \subseteq A)$. By a two-sided ideal or simply an ideal, we mean a non-empty subset of a semigroup S that is both a left and a right ideal of S. A non-empty subset A of S is called an interior ideal of S if $SAS \subseteq A$. A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA \subseteq A$. A non-empty subset A of a semigroup S is called a quasi-ideal of S if $AS \cap SA \subseteq A$. A subsemigroup A of a semigroup S is said to be left (right) bi-quasi ideal of S if $SA \cap ASA \subseteq A(AS \cap ASA \subseteq A)$. A subsemigroup A of a semigroup S is said to be bi-quasi ideal of S if it is both a left bi-quasi and right bi-quasi ideal of S [13].

Let S be a reference set. Then we define hesitant fuzzy set on S in terms of a function $\mathcal{H}$ that when applied to S returns a subset of $[0, 1]$ [6].

For a hesitant fuzzy set $\mathcal{H}$ on S and $x, y, z \in S$, we use the notations $\mathcal{H}_x := \mathcal{H}(x)$, $\mathcal{H}_x^y := \mathcal{H}(x) \cap \mathcal{H}(y)$, and $\mathcal{H}_x^y[z] := \mathcal{H}(x) \cap \mathcal{H}(y) \cap \mathcal{H}(z)$. It is clear that $\mathcal{H}_y^x = \mathcal{H}_x^y$, and

$$\mathcal{H}_x = \mathcal{H}_y \Leftrightarrow \mathcal{H}_x \subseteq \mathcal{H}_y, \mathcal{H}_y \subseteq \mathcal{H}_x$$

for all $x, y \in S$.

Let $\mathcal{H}$ and $\mathcal{G}$ be two hesitant fuzzy sets on S. The hesitant union $\mathcal{H} \sqcup \mathcal{G}$ and hesitant intersection $\mathcal{H} \sqcap \mathcal{G}$ of $\mathcal{H}$ and $\mathcal{G}$ are defined to be hesitant fuzzy sets on S as follows:

$$\mathcal{H} \sqcup \mathcal{G} : S \to \mathcal{P}([0, 1]), x \mapsto \mathcal{H}_x \cup \mathcal{G}_x$$

and

$$\mathcal{H} \sqcap \mathcal{G} : S \to \mathcal{P}([0, 1]), x \mapsto \mathcal{H}_x \cap \mathcal{G}_x,$$

respectively.

For any hesitant fuzzy sets $\mathcal{H}$ and $\mathcal{G}$ on S, we define

$$\mathcal{H} \sqsubseteq \mathcal{G} \text{ if } \mathcal{H}_x \subseteq \mathcal{G}_x \text{ for all } x \in S.$$

The hesitant fuzzy product of $\mathcal{H}$ and $\mathcal{G}$ is defined to be the hesitant fuzzy set $\mathcal{H}\tilde{\circ}\mathcal{G}$ on S which is given by

$$(\mathcal{H}\tilde{\circ}\mathcal{G})_x = \begin{cases} \bigcup\limits_{x=yz} \{\mathcal{H}_y \cap \mathcal{G}_z\} & \text{for some } y, z \in S \text{ such that } x = yz, \\ \emptyset & \text{otherwise.} \end{cases}$$

For a hesitant fuzzy set $\mathcal{H}$ on S and a subset ε of $[0, 1]$, the set

$$\mathcal{S}(\mathcal{H}; \varepsilon) := \{x \in S | \varepsilon \subseteq \mathcal{H}_x\}$$

is called the hesitant level set of $\mathcal{H}$.

For a nonempty subset A of S and $\varepsilon, \delta \in \mathcal{P}([0, 1])$ with $\varepsilon \supsetneq \delta$, define a map $\left[\chi_A^{(\varepsilon,\delta)}\right]$ as follows:

$$\left[\chi_A^{(\varepsilon,\delta)}\right] : S \to \mathcal{P}([0, 1]), x \mapsto \begin{cases} \varepsilon, & x \in A, \\ \delta, & \text{otherwise.} \end{cases}$$

Then $\left[\chi_A^{(\varepsilon,\delta)}\right]$ is a hesitant fuzzy set on S, which is called the (ε, δ)-characteristic hesitant fuzzy set. The hesitant fuzzy set $\left[\chi_S^{(\varepsilon,\delta)}\right]$ is called the (ε, δ)-identity hesitant fuzzy set on S. The (ε, δ)-characteristic hesitant fuzzy set with $\varepsilon = [0, 1]$ and $\delta = \emptyset$ is called the characteristic hesitant fuzzy set, and is denoted by $[\chi_A]$. The (ε, δ)-identity hesitant fuzzy set with $\varepsilon = [0, 1]$ and $\delta = \emptyset$ is called the identity hesitant fuzzy set, and is denoted by $[\chi_S]$.

3. Hesitant Fuzzy Bi-Quasi Ideals.

Definition 3.1. *A hesitant fuzzy set $\mathcal{H}$ on S is called a hesitant fuzzy subsemigroup on S if it satisfies*

$$(\forall x, y \in S)\left(\mathcal{H}_x^y \subseteq \mathcal{H}_{xy}\right).$$

Definition 3.2. *A hesitant fuzzy set $\mathcal{H}$ on S is called a hesitant fuzzy left (resp., right) ideal on S if it satisfies*

$$(\forall x, y \in S)(\mathcal{H}_x^y \supseteq \mathcal{H}_y(resp., \mathcal{H}_x^y \supseteq \mathcal{H}_x)).$$

If a hesitant fuzzy set $\mathcal{H}$ on S is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal on S, we say that $\mathcal{H}$ is a hesitant fuzzy two-sided ideal on S. Obviously, every hesitant fuzzy left (resp., right) ideal on S is a hesitant fuzzy subsemigroup on S. However, the converse is not true in general.

Definition 3.3. *A hesitant fuzzy subsemigroup $\mathcal{H}$ on S is called a hesitant fuzzy left (resp., right) bi-quasi ideal on S if it satisfies the following conditions:*

$$[\chi_S]\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}(resp., \mathcal{H}\tilde{\circ}[\chi_S] \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}).$$

A hesitant fuzzy subsemigroup $\mathcal{H}$ on semigroup S is called a hesitant fuzzy bi-quasi ideal if it is both hesitant fuzzy left bi-quasi ideal and hesitant fuzzy right bi-quasi ideal of S.

Theorem 3.1. *Every hesitant fuzzy left ideal of a semigroup S is a hesitant fuzzy left bi-quasi ideal of S.*

Proof: Let $\mathcal{H}$ be a hesitant fuzzy left ideal of a semigroup S. Let $x \in S$. Then

$$([\chi_S]\tilde{\circ}\mathcal{H})_x = \bigcup_{x=yz}\{[\chi_S]_y \cap \mathcal{H}_z\} = \bigcup_{x=yz}\{\mathcal{H}_z\} \subseteq \bigcup_{x=yz}\{\mathcal{H}_{yz}\} = \bigcup_{x=yz}\{\mathcal{H}_x\} = \mathcal{H}_x.$$

Thus, $[\chi_S]\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}$. Therefore, $\mathcal{H}$ is a hesitant fuzzy left bi-quasi ideal of the semigroup S. □

Theorem 3.2. *Every hesitant fuzzy left ideal of a semigroup S is a hesitant fuzzy right bi-quasi ideal of S.*

Proof: Let $\mathcal{H}$ be a hesitant fuzzy left ideal of a semigroup S. Let $x \in S$. We have $([\chi_S]\tilde{\circ}\mathcal{H})_x \subseteq \mathcal{H}_x$. Then

$$(\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H})_x = \bigcup_{x=abc}\{\mathcal{H}_a \cap ([\chi_S]\tilde{\circ}\mathcal{H})_{bc}\} \subseteq \bigcup_{x=abc}\{\mathcal{H}_a \cap \mathcal{H}_{bc}\} \subseteq \mathcal{H}_x.$$

Thus, $\mathcal{H}\tilde{\circ}[\chi_S] \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}$. Therefore, $\mathcal{H}$ is a hesitant fuzzy right bi-quasi ideal of the semigroup S. □

Theorem 3.3. *Every hesitant fuzzy right ideal of a semigroup S is a hesitant fuzzy right bi-quasi ideal of S.*

Proof: Let $\mathcal{H}$ be a hesitant fuzzy ideal of a semigroup S. Let $x \in S$. Then

$$(\mathcal{H}\tilde{\circ}[\chi_S])_x = \bigcup_{x=yz}\{\mathcal{H}_y \cap [\chi_S]_z\} \subseteq \bigcup_{x=yz}\{\mathcal{H}_{yz}\} = \bigcup_{x=yz}\{\mathcal{H}_x\} = \mathcal{H}_x.$$

Thus, $\mathcal{H}\tilde{\circ}[\chi_S] \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}$. Hence, $\mathcal{H}$ is a hesitant fuzzy right bi-quasi ideal of the semigroup S. □

Corollary 3.1. *Every hesitant fuzzy ideal of a semigroup S is a hesitant fuzzy bi-quasi ideal of S.*

Corollary 3.2. *Every hesitant fuzzy right (left) ideal of a semigroup S is a hesitant fuzzy bi-quasi ideal of S.*

Theorem 3.4. *Let S be a semigroup and $\mathcal{H}$ be a non-empty fuzzy set of S. A fuzzy set $\mathcal{H}$ is a hesitant fuzzy left bi-quasi ideal of a semigroup S if and only if the hesitant level set $\mathcal{S}(\mathcal{H};\varepsilon)$ of $\mathcal{H}$ is a left bi-quasi ideal of a semigroup S for every $\varepsilon \in [0,1]$, where $\mathcal{S}(\mathcal{H};\varepsilon) \neq \emptyset$.*

Proof: Assume that $\mathcal{H}$ is a hesitant fuzzy left bi-quasi ideal of a semigroup S, $\mathcal{S}(\mathcal{H};\varepsilon) \neq \emptyset$, $\varepsilon \in [0,1]$. Let $x \in S\mathcal{S}(\mathcal{H};\varepsilon) \cap \mathcal{S}(\mathcal{H};\varepsilon)S\mathcal{S}(\mathcal{H};\varepsilon)$. Then $x = ba = cde$ where $b, d \in S$ and $a, c, e \in \mathcal{S}(\mathcal{H};\varepsilon)$. Thus, $\varepsilon \subseteq ([\chi_S]\tilde{\circ}\mathcal{H})_x$ and $\varepsilon \subseteq (\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H})_x$ so $\varepsilon \subseteq \mathcal{H}_x$. Hence, $\varepsilon \in \mathcal{H}_x$. Then $x \in \mathcal{S}(\mathcal{H};\varepsilon)$. Therefore, $\mathcal{S}(\mathcal{H};\varepsilon)$ is a left bi-quasi ideal of a semigroup S.

Conversely suppose that $\mathcal{S}(\mathcal{H};\varepsilon)$ is a left bi-quasi ideal of the semigroup S, for all $\varepsilon \in Im(\mathcal{H})$. Let $x, y \in S$. Then $\mathcal{H}_x = \varepsilon_1$, $\mathcal{H}_y = \varepsilon_2$, $\varepsilon_1 \geq \varepsilon_2$. Thus, $x, y \in S(\mathcal{H};\varepsilon)$. We have $S\mathcal{S}(\mathcal{H};\gamma) \cap \mathcal{S}(\mathcal{H};\gamma)S\mathcal{S}(\mathcal{H};\gamma) \subseteq \mathcal{S}(\mathcal{H};\gamma)$, for all $\gamma \in Im(\mathcal{H})$. Suppose $\varepsilon = \min\{Im(\mathcal{H})\}$. Then $S\mathcal{S}(\mathcal{H};\varepsilon) \cap \mathcal{S}(\mathcal{H};\varepsilon)S\mathcal{S}(\mathcal{H};\varepsilon) \subseteq \mathcal{S}(\mathcal{H};\varepsilon)$. Therefore, $[\chi_S]\tilde{\circ}\mathcal{H}\tilde{\circ}\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}$. Hence, $\mathcal{H}$ is a hesitant fuzzy left bi-quasi ideal of a semigroup S. □

Corollary 3.3. *Let S be a semigroup and $\mathcal{H}$ be a non-empty fuzzy set of S. A fuzzy set $\mathcal{H}$ is a hesitant fuzzy right bi-quasi ideal of a semigroup S if and only if the hesitant level set $\mathcal{S}(\mathcal{H};\varepsilon)$ of $\mathcal{H}$ is a right bi-quasi ideal of a semigroup S for every $\varepsilon \in [0,1]$, where $\mathcal{S}(\mathcal{H};\varepsilon) \neq \emptyset$.*

Corollary 3.4. *Let S be a semigroup and $\mathcal{H}$ be a non-empty fuzzy set of S. A fuzzy set $\mathcal{H}$ is a hesitant fuzzy bi-quasi ideal of a semigroup S if and only if the hesitant level set $\mathcal{S}(\mathcal{H};\varepsilon)$ of $\mathcal{H}$ is a bi-quasi ideal of a semigroup S for every $\varepsilon \in [0,1]$, where $\mathcal{S}(\mathcal{H};\varepsilon) \neq \emptyset$.*

Lemma 3.1. *Let $\left[\chi_A^{(\varepsilon,\delta)}\right]$ and $\left[\chi_B^{(\varepsilon,\delta)}\right]$ be (ε,δ)-characteristic hesitant fuzzy sets on S where A and B are nonempty subsets of S. Then the following properties hold.*
1) $\left[\chi_A^{(\varepsilon,\delta)}\right] \sqcap \left[\chi_B^{(\varepsilon,\delta)}\right] = \left[\chi_{A\cap B}^{(\varepsilon,\delta)}\right]$,
2) $\left[\chi_A^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_B^{(\varepsilon,\delta)}\right] = \left[\chi_{AB}^{(\varepsilon,\delta)}\right]$.

Theorem 3.5. *Let I be a non-empty subset of a semigroup S and $\left[\chi_I^{(\varepsilon,\delta)}\right]$ be the (ε,δ)-characteristic hesitant fuzzy sets of I. Then I is a left bi-quasi ideal of a semigroup S if and only if $\left[\chi_I^{(\varepsilon,\delta)}\right]$ is a hesitant fuzzy left bi-quasi ideal of a semigroup S.*

Proof: Suppose I is a left bi-quasi ideal of S. Then I is a subsemigroup of S and $SI \cap ISI \subseteq I$. Obviously, $\left[\chi_I^{(\varepsilon,\delta)}\right]$ is a hesitant fuzzy subsemigroup of S. We have

$$\begin{aligned}
&\left(\left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right] \cap \left[\chi_I^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right]\right)_x \\
&= \left(\left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right]\right)_x \cap \left(\left[\chi_I^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right]\right)_x \\
&= \left[\chi_{SI}^{(\varepsilon,\delta)}\right]_x \cap \left[\chi_{ISI}^{(\varepsilon,\delta)}\right]_x \\
&= \left[\chi_{SI\cap ISI}^{(\varepsilon,\delta)}\right]_x \\
&\subseteq \left[\chi_I^{(\varepsilon,\delta)}\right]_x .
\end{aligned}$$

Thus, $\left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_S^{(\varepsilon,\delta)}\right] \cap \left[\chi_I^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right] \subseteq \left[\chi_I^{(\varepsilon,\delta)}\right]$. Hence, $\left[\chi_I^{(\varepsilon,\delta)}\right]$ is a hesitant fuzzy left bi-quasi ideal of S.

Conversely suppose that $\left[\chi_I^{(\varepsilon,\delta)}\right]$ is a (ε,δ)-characteristic hesitant fuzzy sets of S. Then I is a subsemigroup of S. We have

$$\left(\left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right]\right)_x \cap \left(\left[\chi_I^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_I^{(\varepsilon,\delta)}\right]\right)_x \subseteq \left[\chi_I^{(\varepsilon,\delta)}\right]_x$$
$$\Rightarrow \left[\chi_{SI}^{(\varepsilon,\delta)}\right]_x \cap \left[\chi_{ISI}^{(\varepsilon,\delta)}\right]_x \subseteq \left[\chi_I^{(\varepsilon,\delta)}\right]_x$$
$$\Rightarrow \left[\chi_{SI\cap ISI}^{(\varepsilon,\delta)}\right]_x \subseteq \left[\chi_I^{(\varepsilon,\delta)}\right]_x.$$

We have $SI \cap ISI \subseteq I$. Hence, I is a left bi-quasi ideal of a semigroup S. □

Theorem 3.6. *Let I be a non-empty subset of a semigroup S and $\left[\chi_I^{(\varepsilon,\delta)}\right]$ be the (ε,δ)-characteristic hesitant fuzzy sets of I. Then I is a right bi-quasi ideal of a semigroup S if and only if $\left[\chi_I^{(\varepsilon,\delta)}\right]$ is a hesitant fuzzy right bi-quasi ideal of a semigroup S.*

Proof: It follows by Theorem 3.5. □

Corollary 3.5. *Let I be a non-empty subset of a semigroup S and $\left[\chi_I^{(\varepsilon,\delta)}\right]$ be the (ε,δ)-characteristic hesitant fuzzy sets of I. Then I is a bi-quasi ideal of a semigroup S if and only if $\left[\chi_I^{(\varepsilon,\delta)}\right]$ is a hesitant fuzzy bi-quasi ideal of a semigroup S.*

Theorem 3.7. *If $\mathcal{H}$ and $\mathcal{G}$ are hesitant fuzzy bi-quasi ideals of a semigroup S, then $\mathcal{H} \sqcap \mathcal{G}$ is a hesitant fuzzy left bi-quasi ideal of a semigroup S.*

Proof: Let $\mathcal{H}$ and $\mathcal{G}$ be hesitant fuzzy bi-quasi ideals of a semigroup S. Then

$$\begin{aligned}
([\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{G})_x &= \bigcup_{x=ab}\{[\chi_S]_a \cap (\mathcal{H}\cap\mathcal{G})_b\} \\
&= \bigcup_{x=ab}\{[\chi_S]_a \cap \mathcal{H}_b \cap \mathcal{G}_b\} \\
&= \bigcup_{x=ab}\{\{[\chi_S]_a \cap \mathcal{H}_b\} \cap \{[\chi_S]_a \cap \mathcal{G}_b\}\} \\
&= \bigcup_{x=ab}\{[\chi_S]_a \cap \mathcal{H}_b\} \cap \bigcup_{x=ab}\{[\chi_S]_a \cap \mathcal{G}_b\} \\
&= ([\chi_S]\tilde{\circ}\mathcal{H})_x \cap ([\chi_S]\tilde{\circ}\mathcal{G})_x \\
&= ([\chi_S]\tilde{\circ}\mathcal{H} \cap [\chi_S]\tilde{\circ}\mathcal{G})_x.
\end{aligned}$$

Therefore, $[\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{G} = [\chi_S]\tilde{\circ}\mathcal{H}\cap[\chi_S]\tilde{\circ}\mathcal{G}$.

$$\begin{aligned}
(\mathcal{H}\cap\mathcal{G}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{G})_x &= \bigcup_{x=abc}\{(\mathcal{H}\cap\mathcal{G})_a \cap ([\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{G})_{bc}\} \\
&= \bigcup_{x=abc}\{(\mathcal{H}\cap\mathcal{G})_a \cap \{([\chi_S]\circ\mathcal{H}\cap[\chi_S]\tilde{\circ}\mathcal{G})_{bc}\}\} \\
&= \bigcup_{x=abc}\{(\mathcal{H}\cap\mathcal{G})_a \cap \{([\chi_S]\tilde{\circ}\mathcal{H})_{bc} \cap ([\chi_S]\tilde{\circ}\mathcal{G})_{bc}\}\} \\
&= \bigcup_{x=abc}\{\{\mathcal{H}_a \cap ([\chi_S]\tilde{\circ}\mathcal{H})_{bc}\} \cap \{\mathcal{G}_a \cap ([\chi_S]\circ\mathcal{G})_{bc}\}\} \\
&= (\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H})_x \cap (\mathcal{G}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{G})_x \\
&= (\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{G}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{G})_x.
\end{aligned}$$

Therefore, $\mathcal{H}\cap\mathcal{G}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{G} = \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{G}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{G}$. Then $[\chi_S]\tilde{\circ}(\mathcal{H}\cap\mathcal{G})\cap(\mathcal{H}\cap\mathcal{G})\tilde{\circ}[\chi_S]\tilde{\circ}(\mathcal{H}\cap\mathcal{G}) = [\chi_S]\tilde{\circ}\mathcal{H}\cap\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}\cap[\chi_S]\tilde{\circ}\mathcal{G}\cap\mathcal{G}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{G} \subseteq \mathcal{H}\cap\mathcal{G}$. Hence, $\mathcal{H}\sqcap\mathcal{G}$ is a hesitant fuzzy left bi-quasi ideal of a semigroup S. □

The following results are immediate consequences of Theorem 3.7.

Corollary 3.6. *If $\mathcal{H}$ and $\mathcal{G}$ are hesitant fuzzy bi-quasi ideals of a semigroup S, then $\mathcal{H} \sqcap \mathcal{G}$ is a hesitant fuzzy right bi-quasi ideal of a semigroup S.*

Corollary 3.7. *If $\mathcal{H}$ and $\mathcal{G}$ are hesitant fuzzy bi-quasi ideals of a semigroup S, then $\mathcal{H} \sqcap \mathcal{G}$ is a hesitant fuzzy bi-quasi ideal of a semigroup S.*

Theorem 3.8. *If $\mathcal{H}$ and $\mathcal{G}$ are hesitant fuzzy right ideals and a hesitant fuzzy left ideal of a semigroup S, respectively, then $\mathcal{H} \sqcap \mathcal{G}$ is a hesitant fuzzy left bi-quasi ideal of a semigroup S.*

Corollary 3.8. *If $\mathcal{H}$ and $\mathcal{G}$ are hesitant fuzzy right ideals and a hesitant fuzzy left ideal of a semigroup S, respectively, then $\mathcal{H} \sqcap \mathcal{G}$ is a hesitant fuzzy right bi-quasi ideal of a semigroup S.*

Corollary 3.9. *If $\mathcal{H}$ and $\mathcal{G}$ are hesitant fuzzy right ideals and a hesitant fuzzy left ideal of a semigroup S, respectively, then $\mathcal{H} \sqcap \mathcal{G}$ is a hesitant fuzzy bi-quasi ideal of a semigroup S.*

Definition 3.4. *A semigroup S is called regular if for all $a \in S$ there exists $x \in S$ such that $a = axa$.*

Definition 3.5. *A hesitant fuzzy subsemigroup $\mathcal{H}$ on S is called a hesitant fuzzy quasi-ideal on S if it satisfies the following conditions:*

$$\mathcal{H}\tilde{\circ}[\chi_S] \cap [\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}.$$

Theorem 3.9. *If $\mathcal{H}$ is a hesitant fuzzy quasi-ideal of a regular semigroup S, then $\mathcal{H}$ is a hesitant fuzzy ideal of a semigroup S.*

Proof: Assume that $\mathcal{H}$ is a hesitant fuzzy quasi-ideal of S and let $x, y \in S$. Then

$$\begin{aligned} \mathcal{H}_{xy} &\supseteq (\mathcal{H}\tilde{\circ}[\chi_S])_{xy} \cap ([\chi_S]\tilde{\circ}\mathcal{H})_{xy} \\ &= \bigcup_{xy=ab} \{\mathcal{H}_a \cap [\chi_S]_b\} \cap \bigcup_{xy=ij} \{[\chi_S]_i \cap \mathcal{H}_j\} \\ &\supseteq \mathcal{H}_x \cap [\chi_S]_y \cap [\chi_S]_x \cap \mathcal{H}_y \\ &= \mathcal{H}_x \cap \mathcal{H}_y. \end{aligned}$$

Thus, $\mathcal{H}_{xy} \supseteq \mathcal{H}_x \cap \mathcal{H}_y$. Hence, $\mathcal{H}$ is a hesitant fuzzy subsemigroup of S. Let $x, y, z \in S$. Then

$$\begin{aligned} \mathcal{H}_{xyz} &\supseteq (\mathcal{H}\tilde{\circ}[\chi_S])_{xyz} \cap ([\chi_S]\tilde{\circ}\mathcal{H})_{xyz} \\ &= \bigcup_{xyz=ab} \{\mathcal{H}_a \cap [\chi_S]_b\} \cap \bigcup_{xyz=ij} \{[\chi_S]_i \cap \mathcal{H}_j\} \\ &\supseteq \mathcal{H}_x \cap [\chi_S]_{yz} \cap [\chi_S]_{xy} \cap \mathcal{H}_z \\ &= \mathcal{H}_x \cap \mathcal{H}_z. \end{aligned}$$

Thus, $\mathcal{H}_{xyz} \supseteq \mathcal{H}_x \cap \mathcal{H}_z$. Hence, $\mathcal{H}$ is a hesitant fuzzy bi-ideal of S. Since S is regular, $\mathcal{H}$ is a hesitant fuzzy bi-ideal of S and $x, y \in S$ we have $xy \in (xSx)S \subseteq xSx$. Thus, there exists $k \in S$ such that $xy = xkx$. So

$$\mathcal{H}_{xy} = \mathcal{H}_{xkx} \supseteq \mathcal{H}_x \cap \mathcal{H}_x = \mathcal{H}_x.$$

Thus, $\mathcal{H}$ is a hesitant fuzzy right ideal of S. Similarly, we can show that $\mathcal{H}_{xy} \supseteq \mathcal{H}_y$. Thus, $\mathcal{H}$ is a hesitant fuzzy left ideal of S. Hence, $\mathcal{H}$ is a hesitant fuzzy ideal of S. □

Theorem 3.10. *Let S be a regular semigroup. Then $\mathcal{H}$ is a hesitant fuzzy left bi-quasi ideal of S if and only if $\mathcal{H}$ is a hesitant fuzzy quasi-ideal of S.*

Proof: Let $\mathcal{H}$ be a hesitant fuzzy left bi-quasi ideal of S and $x \in S$. Thus, $([\chi_S]\tilde{\circ}\mathcal{H})_x \cap (\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H})_x \subseteq \mathcal{H}_x$. Suppose $([\chi_S]\tilde{\circ}\mathcal{H})_x \supseteq \mathcal{H}_x$. Since S is regular, there exists $y \in S$ such that $x = xyx$. Then

$$(\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H})_x = \bigcup_{x=xyx} \{\mathcal{H}_{xy} \cap ([\chi_S]\tilde{\circ}\mathcal{H})_x\} \supseteq \bigcup_{x=xyx} \{\mathcal{H}_x \cap \mathcal{H}_x\} = \mathcal{H}_x,$$

which is a contradiction. Therefore, $\mathcal{H}$ is a hesitant fuzzy quasi-ideal of S. By Theorem 3.9, converse is true. □

Corollary 3.10. *Let S be a regular semigroup. Then $\mathcal{H}$ is a hesitant fuzzy right bi-quasi ideal of S if and only if $\mathcal{H}$ is a hesitant fuzzy quasi-ideal of S.*

Corollary 3.11. *Let S be a regular semigroup. Then $\mathcal{H}$ is a hesitant fuzzy bi-quasi ideal of S if and only if $\mathcal{H}$ is a hesitant fuzzy quasi-ideal of S.*

Theorem 3.11. *Let S be a semigroup. S is a regular semigroup if and only if $B = SB \cap BSB$, for every bi-quasi ideal of S.*

Theorem 3.12. *Let S be a semigroup. Then S is a regular semigroup if and only if $\mathcal{H} = [\chi_S]\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}$, for any hesitant fuzzy left bi-quasi ideal of a semigroup S.*

Proof: Let $\mathcal{H}$ be a hesitant fuzzy left bi-quasi ideal of the regular semigroup S. Then $[\chi_S]\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H} \subseteq \mathcal{H}$. Let $x \in S$. Since S is regular, there exists $a \in S$ such that $x = xax$. Thus,

$$\begin{aligned}(\mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H})_x &= \bigcup_{x=xax} \{\mathcal{H}_x \cap ([\chi_S]\tilde{\circ}\mathcal{H})_{ax}\} \\ &= \bigcup_{x=xax} \left\{\mathcal{H}_x \cap \bigcup_{ax=yz} \{[\chi_S]_y \cap \mathcal{H}_z\}\right\} \\ &\supseteq \bigcup_{x=xax} \{\mathcal{H}_x \cap \mathcal{H}_x\} \\ &= \mathcal{H}_x.\end{aligned}$$

Therefore, $\mathcal{H} = [\chi_S]\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}$.

Conversely suppose that let B be a left bi-quasi ideal of a semigroup S. Then by Theorem 3.5, $\left[\chi_B^{(\varepsilon,\delta)}\right]$ is (ε, δ)-characteristic hesitant fuzzy sets of the semigroup S. Thus,

$$\begin{aligned}\left[\chi_B^{(\varepsilon,\delta)}\right]_x &= \left(\left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_B^{(\varepsilon,\delta)}\right]\right)_x \cap \left(\left[\chi_B^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_S^{(\varepsilon,\delta)}\right] \tilde{\circ} \left[\chi_B^{(\varepsilon,\delta)}\right]\right)_x \\ &= \left[\chi_{SB}^{(\varepsilon,\delta)}\right]_x \cap \left[\chi_{BSB}^{(\varepsilon,\delta)}\right]_x \\ &= \left[\chi_{SB\cap BSB}^{(\varepsilon,\delta)}\right]_x.\end{aligned}$$

Therefore, $B = SB \cap BSB$. By Theorem 3.11, S is a regular semigroup. □

Corollary 3.12. *Let S be a semigroup. Then S is a regular semigroup if and only if $\mathcal{H} = [\chi_S]\tilde{\circ}\ \mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}$, for any hesitant fuzzy right bi-quasi ideal of a semigroup S.*

Corollary 3.13. *Let S be a semigroup. Then S is a regular semigroup if and only if $\mathcal{H} = [\chi_S]\tilde{\circ}\ \mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}$ or $\mathcal{H} = \mathcal{H}\tilde{\circ}[\chi_S] \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}$, for any hesitant fuzzy bi-quasi ideal of a semigroup S.*

Theorem 3.13. *Let S be a semigroup. Then S is a regular semigroup if and only if $\mathcal{H} \cap \mathcal{G} \subseteq \mathcal{G}\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}\mathcal{G}\tilde{\circ}\mathcal{H}$, for every hesitant fuzzy left bi-quasi ideal $\mathcal{H}$ and every hesitant fuzzy ideal $\mathcal{G}$ of a semigroup S.*

Proof: Let S be a regular semigroup and $x \in S$. Then there exists $y \in S$ such that $x = xyx$.

$$\begin{aligned}(\mathcal{H}\tilde{\circ}\mathcal{G}\tilde{\circ}\mathcal{H})_x &= \bigcup_{x=xyx} \{(\mathcal{H}\tilde{\circ}\mathcal{G})_{xy} \cap \mathcal{H}_x\} \\ &= \bigcup_{x=xyx} \left\{\bigcup_{xy=xyxy} \{\mathcal{H}_x \cap \mathcal{G}_{yxy}\} \cap \mathcal{H}_x\right\}\end{aligned}$$

$$\supseteq \{\mathcal{H}_x \cap \mathcal{G}_x\} \cap \mathcal{H}_x$$
$$= \mathcal{H}_x \cap \mathcal{G}_x$$
$$= (\mathcal{H} \cap \mathcal{G})_x.$$

and

$$(\mathcal{G} \circ \mathcal{H})_x = \bigcup_{x=xyx} \{\mathcal{G}_{xy} \cap \mathcal{H}_x\} \supseteq \{\mathcal{G}_x \cap \mathcal{H}_x\} = (\mathcal{G} \cap \mathcal{H})_x.$$

Hence, $\mathcal{H} \cap \mathcal{G} \subseteq \mathcal{G}\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}\mathcal{G}\tilde{\circ}\mathcal{H}$.

Conversely suppose that the condition holds. Let $\mathcal{H}$ be a hesitant fuzzy left bi-quasi ideal. We have $\mathcal{H} \cap [\chi_S] \subseteq [\chi_S]\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}$ implies that $\mathcal{H} \subseteq [\chi_S]\tilde{\circ}\mathcal{H}\tilde{\circ} \cap \mathcal{H}\tilde{\circ}[\chi_S]\tilde{\circ}\mathcal{H}$. By Theorem 3.12, S is a regular semigroup. $\square$

Corollary 3.14. *Let S be a semigroup. Then S is a regular semigroup if and only if $\mathcal{H} \cap \mathcal{G} \subseteq \mathcal{G}\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}\mathcal{G}\tilde{\circ}\mathcal{H}$, for every hesitant fuzzy right bi-quasi ideal $\mathcal{H}$ and every hesitant fuzzy ideal $\mathcal{G}$ of a semigroup S.*

Corollary 3.15. *Let S be a semigroup. Then S is a regular semigroup if and only if $\mathcal{H} \cap \mathcal{G} \subseteq \mathcal{G}\tilde{\circ}\mathcal{H} \cap \mathcal{H}\tilde{\circ}\mathcal{G}\tilde{\circ}\mathcal{H}$, for every hesitant fuzzy bi-quasi ideal $\mathcal{H}$ and every hesitant fuzzy ideal $\mathcal{G}$ of a semigroup S.*

4. **Conclusion.** In this paper, we study concept of a hesitant fuzzy bi-quasi ideal of semigroups and discuss properties of hesitant fuzzy bi-quasi ideals of semigroups. Finally, we characterize the regular semigroup in terms of a hesitant fuzzy bi-quasi ideal of a semigroup. In the future work, we can study hesitant fuzzy bi-interior ideals of semigroups or algebraic structures.

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