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A DUAL RELAXATION BOUND TO A MIXED LOGIT PRICING MODEL

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Abstract. *Determining optimal pricing for multiple substitutable products across mixed customer segments is a fundamental objective for businesses aiming to sustain competitiveness. This paper focuses on the study of optimal pricing strategies for multiple substitutable products within a fixed time horizon, while accounting for the heterogeneity of customer preferences across different segments. To model this heterogeneity, a mixed logit model is utilized. Pricing optimization with the overlapping multiple segments is a difficult nonconvex maximization problem. We develop a binary fractional programming model that computes relaxation upper bound for the mixed logit pricing problem by disaggregating and discretizing price vectors. We next develop an effective Lagrangian dual formulation that uses simple Newton search to compute dual function. For randomly generated problem instances, we observed that the proposed model achieves within one percent of the optimal value for small sized problem, and it consistently generates tight upper bound for bigger sized problems within 500 CPU seconds.*
Keywords: Mixed logit pricing model, Multinomial logit demand function, Revenue management, Fractional programming, Lagrangian dual

1. Introduction.

Numerous online retailers with a presence in both virtual and physical stores spanning extensive geographical regions encounter challenges in determining appropriate pricing strategies for their diverse product lines. In order to enhance revenue and profitability, these companies heavily rely on conventional pricing software solutions. However, in order to effectively leverage the variations in customer preferences, pricing decisions must take account of the unique characteristics of customer utility. In recent omnichannel retailing environment, increasing competition between online, offline, competitor's channels, product bundle pricing, and new order fulfillment option such as buy online pick up in store necessitate accurate customer preference modeling [1-4]. The multinomial logit (MNL) model has conventionally served as a valuable analytical tool for estimating customer choice preferences. Moreover, to adequately capture the heterogeneity within customer groups across different segments or geographic zones, the multi-segment mixed logit model has gained widespread adoption [5,6].

For unconstrained nested MNL model, the optimal revenue value is characterized as a fixed point of a nonlinear equation. Thus, for a single segment pricing problem, if the prices in that segment are independent to other segments, optimal revenue prices can be computed using one-dimensional search algorithm [7]. For an MNL pricing problem with linear constraints, it has been observed that the total revenue function as the sum of price-weighted expected demand is not in general concave or quasi-concave function of prices [8]. Instead of price, market shares can be used as decision variables and resulting formulation can be solved as a traditional convex optimization problem. For an MNL mixed logit problem, near optimal approximation solution is computed using modified

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attraction function when the attraction function is locally Lipschitz continuous [9]. In the market share variable reformulation, if the no purchase probabilities for each segment are fixed as constraints, the Lagrangian dual of the resulting formulation becomes simple convex minimization problem. Using this idea, a branch-and-bound approximation algorithm in the hypercube is developed [11]. For a multi-segment MNL model, when most of the price vectors are segment-dependent with a few segment-independent price variables present, a mixed integer programming model is developed to address pricing optimization in the case of discrete prices, where some prices are segment-independent. Their model considers scenarios where a retailer operates both online and brick-and-mortar stores across multiple zones, and customers in different zones possess a mixture of zone-specific and price-specific attributes [12].

Due to the inherent non-convex nature of the mixed logit pricing problem, obtaining a global optimal solution necessitates the utilization of approximate or exact algorithms, which rely on computing a tightly bounded upper limit for the optimal value of the problem. In this paper, we develop a binary fractional programming model that computes relaxation upper bound for the mixed logit pricing problem by disaggregating and discretizing price vectors. We next develop an effective Lagrangian dual formulation that is separable with respect to segments and its dual function is efficiently computed by a Newton line search.

In Section 2, we describe a mixed logit pricing problem formulated as a general nonlinear programming problem. Section 3 describes a disaggregated relaxation model, its Lagrangian dual formulation, and solution algorithms. Computational experiences of the nonlinear mixed logit pricing problem, its approximation, and the proposed model are provided in Section 4. Finally, Section 5 concludes the paper.

2. Problem Statement and Preliminaries.

There are $L = \{1, \ldots, c\}$ segments of customers and the probability of a customer belonging to segment l is given as Γ_l; thus, $\sum_{l \in L} \Gamma_l = 1$. Let $N = \{1, \ldots, n\}$ denote a set of products, and we assume that a customer may purchase any products offered, but her choice probability is dependent on particular segment she belongs. An attractiveness of product i in segment l is denoted by a strictly decreasing twice differentiable function $f_j^l(x_j)$. In MNL demand function, $f_j^l(x_j) = e^{a_{lj} - b_j x_j}$, where a_{lj} is price-independent component and b_j represents price-sensitive components. Attraction functions used in other demand models are: in linear attraction demand model, the attraction function is given as $f_i(x_i) = \alpha_i - \beta_i x_i$, $\alpha_i, \beta_i > 0$, while in multiplicative competitive interaction (MCI) demand model, $f_i(x_i) = \alpha_i x_i^{-\beta_i}$, $\alpha_i > 0$ and $\beta_i > 1$.

Let a_i denote the unit profit of product i and let $q_{il}(\mathbf{x})$ denote the probability that customer in segment l purchases product i given price vector $\mathbf{x} = (x_1, \ldots, x_n)$. Mixed logit profit maximization problem is formulated as the following nonlinear programming problem.

$$\text{(P)} \quad \max \sum_{i \in N} \sum_{l \in L} \Gamma_l a_i x_i q_{il}(\mathbf{x}) \tag{1}$$

$$\text{s.t.} \quad \sum_{i \in N} \sum_{l \in L} A_{mi} \Gamma_l q_{il}(\mathbf{x}) \le u_m \quad m \in M \triangleq \{1, \ldots, s\} \tag{2}$$

$$l_i \le x_i \le u_i \quad i \in N. \tag{3}$$

The demand probability $q_{il}(\mathbf{x})$ of product i and no purchase probability $q_{0l}(\mathbf{x})$ in segment l is

$$q_{il}(\mathbf{x}) = \frac{f_i^l(x_i)}{1 + \sum_{j \in N} f_j^l(x_j)}, \quad q_{0l}(\mathbf{x}) = \frac{1}{1 + \sum_{j \in N} f_j^l(x_j)}. \tag{4}$$

The objective function (1) denotes total expected profit when the prices are $\mathbf{x} = (x_1, \ldots, x_n)$.

Note that demand of product i in segment l depends on the entire price vector $\mathbf{x}$. Constraints (2) capture retailer's operational constraints and constraints (3) denote the lower and upper bounds for the prices. The modeling of constraints pertaining to the demand function and price vectors encompasses discrete pricing, volume or sales goal constraints, as well as a price monotonicity constraint that mandates a specified percentage difference in price between a particular product and a substitutable similar product. Additionally, price bound constraints are employed to define the allowable percentage difference relative to historic price or competitor's price.

3. **Main Results.** The objective function in problem (P) is not a concave, or pseudo-concave function of prices and with linear constraints it has many local maxima. It is demonstrated with two-product, three linear constraints, the level set of the profit function (1) is not a convex set [8,9]. For unconstrained case considered in [11], for certain values of no purchase probability, the convex subproblems are frequently infeasible or unbounded, so there needs to be a robust solution approach to problem (P).

In this section, we develop a relaxation of (P) and its Lagrangian dual formulation. Note that Lagrangian dual function is usually a convex function even though the primal objective function is not a concave function. To obtain a relaxation bound to (P), the price vector x_i is replaced with the segment dependent price x_{il}. Using x_{il}, the objective function and constraints of (P) are reformulated as

$$\begin{aligned} \text{(RP)}\quad \max & \sum_{i \in N} a_i \sum_{l \in L} \Gamma_l x_{il} q_{il}(x_{il}) \\ \text{s.t.} & \sum_{i \in N} A_{mi} \sum_{l \in L} \Gamma_l q_{il}(x_{il}) \le u_m \quad m \in M \\ & l_i \le x_{il} \le u_i, \quad i \in N. \end{aligned}$$

From a feasible solution $\bar{\mathbf{x}}$ of (P), setting $\bar{x}_{il} = \bar{x}_i$, $i \in N$, $l \in L$, $(\bar{x}_{il})$ are a feasible solution to (RP). Also, we restrict $x_{il} \in I_{il}$ where I_{il} is a finite set containing possible values of price x_{il}. Restriction of price values to a discrete set I_{il} is the result of common observation in ticket price determination process in retailers. Usually the ticket prices are discrete, often end with magic number endings (such as \$.99), and there are historic lower and upper bounds to sticker prices. So we can represent each price x_{il} as

$$\begin{aligned} & x_{il} = \sum_{k \in I_{il}} p_{ilk} z_{ilk} \\ & \sum_{k \in I_{il}} z_{ilk} = 1 \quad i \in N, l \in L \\ & z_{ilk} \in \{0, 1\} \quad i \in N, l \in L, k \in I_{il}. \end{aligned}$$

Let $q_{ilk} = a_i \Gamma_l p_{ilk} f_i^l(p_{ilk})$, $\alpha_{milk} = A_{mi} \Gamma_l f_i^l(p_{ilk})$, $r_{jlk} = f_j^l(p_{jlk})$, and we can reformulate (RP) as the following binary fractional programming problem.

$$\text{(RDP)}\quad \max \frac{\sum_{l \in L} \sum_{i \in N} \sum_{k \in I_{il}} q_{ilk} z_{ilk}}{1 + \sum_{j \in N} \sum_{k \in I_{jl}} r_{jlk} z_{jlk}} \tag{5}$$

$$\text{s.t.}\quad \frac{\sum_l \sum_i \sum_k \alpha_{milk} z_{ilk}}{1 + \sum_j \sum_k r_{jlk} z_{jlk}} \le u_m \quad m \in M \tag{6}$$

$$\sum_{k \in I_{il}} z_{ilk} = 1 \quad i \in N, l \in L \tag{7}$$

$$z_{ilk} \in \{0, 1\} \quad i \in N, l \in L, k \in I_{il}. \tag{8}$$

Formulation (RDP) can be further reformulated as a linear integer programming model by the standard Charnes and Cooper variable substitution. In this paper, we try to solve directly the binary fractional programming formulation (RDP). For problem (RDP), define dual function $\phi(\lambda, z)$ as

$$\phi(\lambda, z) = \max_{z \in Z} \left\{ \frac{\sum_{l \in L} \sum_{i \in N} \sum_{k \in I_{il}} q_{ilk} z_{ik} - \sum_{l \in L} \sum_{i \in N} \sum_{k \in I_{il}} \sum_{m \in M} \lambda_m \alpha_{milk} z_{ik}}{1 + \sum_{j \in N} \sum_{k \in I_{jl}} r_{jlk} z_{jk}} \right\}$$

$$= \sum_{l \in L} \phi_l(\lambda, z), \text{ where } \phi_l(\lambda, z) = \max_{z \in Z} \left\{ \frac{\sum_{i \in N} \sum_{k \in I_{il}} \left(q_{ilk} - \sum_{m \in M} \lambda_m \alpha_{milk}\right) z_{ilk}}{1 + \sum_{j \in N} \sum_{k \in I_{jl}} r_{jlk} z_{jlk}} \right\}.$$

and $Z = \left\{\sum_{k \in I_{il}} z_{ilk} = 1, \forall i, z_{ilk} \in \{0, 1\}, \forall i, l, k\right\}$. Note that dual function $\phi(\lambda, z)$ and set Z are separable with respect to segment l. Then the Lagrangian dual formulation of problem (RDP) is the following minimization problem.

$$\text{(LD)} \ \min_{\lambda \geq 0} \sum_{l \in L} \phi_l(\lambda, z) \tag{9}$$

Dual function $\phi_l(\lambda, z)$ is computed using the following algorithm for the fractional programming problem. To simplify the presentation, we consider following linear fractional programming problem.

$$\text{(LFP)} \ \max \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} z_{ij}}{1 + \sum_{s=1}^{m} \sum_{t=1}^{n} d_{st} z_{st}} \tag{10}$$

$$\text{s.t.} \ \sum_{j=1}^{n} z_{ij} = 1 \quad i = 1, \ldots, m \tag{11}$$

$$z_{ij} \in \{0, 1\} \quad i = 1, \ldots, m, \ j = 1, \ldots, n. \tag{12}$$

In the denominator of Equation (10), parameter d_{st} corresponds to some attraction function value r_{jlk} in (RDP) and it is strictly positive. For solving (LFP), we can apply Dinkelbach method. Define

$$F(\alpha) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{ij} - \alpha d_{ij}) z_{ij} - \alpha.$$

Then, the optimal value of LFP is obtained by solving the following one-dimensional problem with respect to α

$$\min_{\alpha} \{\alpha | F(\alpha) \leq 0\} \tag{13}$$

Note that for a given value of α, computation of $F(\alpha)$ with constraints (11), (12) is computed as

$$F(\alpha) = \sum_{i=1}^{m} \max_{j} \{c_{ij} - \alpha d_{ij}\} - \alpha. \tag{14}$$

We use Newton search algorithm to find the optimal α in (13). Note that since $F(\alpha)$ is a sum of pointwise maximum of affine functions, $F(\alpha)$ is a convex decreasing function of α and z_{ilk} in (RDP) automatically satisfy constraints (7), (8). For the minimization problem (LD), we can apply any convex function minimization algorithm. For our computation, we apply a deflected subgradient minimization algorithm [13].

4. **Computational Results.** Results for the (RDP) formulations are given in Tables 1 and 2. In this section, we compare the objective bound and approximation errors for the formulation (P), its local approximations and (RDP). For the nonlinear programming models, we use AMPL LOQO commercial solver on a PC with 32 GB memory. LOQO solver uses a primal-dual interior point algorithm for sequential quadratic programming. For relaxation of the Lagrangian dual problem, we develop a custom Python code. In the

Newton line search algorithm, we observed that the optimal value of α in (13) is obtained within 10 iterations of Newton search with tolerance set at 10^{-7}.

To introduce a local approximation to problem (P), we introduce the following notation. Details of the approximation proof can be found in [8]. When the attraction function is locally Lipschitz at point $\mathbf{x}_0$, define segment weight parameter γ_l as

$$\gamma_l = \frac{\Gamma_l q_{0l}(\mathbf{x}_0)}{\sum_{l \in L} \Gamma_l q_{0l}(\mathbf{x}_0)} \tag{15}$$

where $q_{0l}(\mathbf{x}_0)$ is no purchase probability of segment l in Equation (4). Using γ_l instead of Γ_l, the attraction function and demand functions in the neighborhood $B(\mathbf{x}_0, \epsilon)$ of $\mathbf{x}_0$ are approximated as

$$\bar{f}_i(x_i) = \sum_{l \in L} \gamma_l f_i^l(x_i), \quad \bar{q}_{il}(\mathbf{x}) = \frac{\bar{f}_i(x_i)}{1 + \sum_{j \in N} \bar{f}_j(x_j)} = \frac{\sum_{l \in L} \gamma_l f_i^l(x_i)}{1 + \sum_{j \in N} \sum_{l \in L} \gamma_l f_j^l(x_j)}.$$

We denote (P') as the formulation (P) using demand function $\bar{q}_{il}(\mathbf{x}_0)$ and $\left(\bar{\mathrm{P}}\right)$ and $\left(\bar{\mathrm{P}}'\right)$ as the problem (P) and (P') without ball constraints $\mathbf{x} \in B(\mathbf{x_0}, \epsilon)$. Thus, in $\left(\bar{\mathrm{P}}\right)$ and $\left(\bar{\mathrm{P}}'\right)$, price vectors $\mathbf{x}$ can be outside the ball $B(\mathbf{x}_0, \epsilon)$. In Table 1, we compare the objective bound between (P), (P'), $\left(\bar{\mathrm{P}}\right)$ and $\left(\bar{\mathrm{P}}'\right)$. Reference point $\mathbf{x}_0$ in (15) is suggested to satisfy $q_{il}(\mathbf{x}_0) = \bar{q}_{il}(\mathbf{x}_0) = \frac{1}{n+1}$. In Table 1, Ratio 1, Ratio 2, and Ratio 3 denote percentage differences $|z(\mathrm{P}') - z(\mathrm{P})|/z(\mathrm{P})$, $\left|z\left(\bar{\mathrm{P}}\right) - z(\mathrm{P})\right|/z(\mathrm{P})$, $\left|z\left(\bar{\mathrm{P}}'\right) - z(\mathrm{P}')\right|/z(\mathrm{P}')$, respectively, where $z(\mathrm{P})$ denotes the objective value of formulation (P). From Ratio 1, we observe that near reference point $\mathbf{x}_0$, the objective value difference is within 2.4% of (P). When we remove the ball constraints, Ratio 2 shows that the objective value differences are ranging between 0.66% to 12.7%. Ratio 3 shows that the objective value differences between (P') and $\left(\bar{\mathrm{P}}'\right)$ are ranging from 0.44% to 9.54%.

Table 1. Comparison of nonlinear formulations

Products	Segments	Constraints	Iterations				Ratio 1	Ratio 2	Ratio 3
			(P)	(P')	$\left(\bar{\mathrm{P}}\right)$	$\left(\bar{\mathrm{P}}'\right)$	(%)	(%)	(%)
32	8	128	50	50	25	26	0.34	0.54	0.44
32	8	128	21	21	14	14	0.51	0.66	0.66
32	8	128	50	50	23	22	0.20	1.22	1.25
32	8	128	50	50	23	23	0.14	1.17	1.40
32	8	128	50	50	22	22	0.49	0.97	0.81
64	8	128	27	28	18	19	2.93	12.61	9.54
64	8	128	20	22	15	16	0.07	3.78	3.80
64	8	128	21	22	15	15	0.23	2.92	2.99
64	8	128	20	20	16	16	0.05	3.15	3.18
64	8	128	19	19	15	16	0.15	1.82	1.85

Table 2 shows the CPU times of LOQO optimizer and (RDP), and objective value ratio between (RDP) and (P). For LOQO, iteration limit is set 50 and time limit is set at 1,000 seconds. For (RDP), discrete value set I_{il} is assumed as 40 points in the interval $[x_{0j} \pm 0.2x_{0j}]$ for each $j \in N$. For problems 1-10 in Table 2, CPU times of LOQO are within 570 seconds and CPU times of (RDP) are all within 57 seconds. Also, the (RDP)/(P) ratio shows that for these 10 problems, (RDP) overestimates the objective value of (P) within 7.8%. For the first 5 problems, (RDP)/(P) ratio is within 1%. Therefore, for small sized problem (RDP) provides tight bound to (P).

TABLE 2. Optimal value ratios and CPU times of RDP formulation

	Products	Segments	Constraints	CPU time (s)	(RDP)/(P)	CPU time (s)
1	32	8	128	75	100.23	27
2	32	8	128	65	100.65	26
3	32	8	128	67	100.95	24
4	32	8	128	63	100.29	27
5	32	8	128	74	100.52	25
6	64	8	128	564	107.78	54
7	64	8	128	323	102.34	55
8	64	8	128	269	102.00	54
9	64	8	128	232	101.98	57
10	64	8	128	282	101.29	55
11	128	8	256	1044	109.45*	217
12	128	8	256	1042	107.07*	216
13	128	8	256	1046	108.30*	216
14	128	8	256	1049	109.56*	217
15	128	8	256	1047	107.92*	217
16	256	8	128	1601	111.30*	220
17	256	16	128	3906	116.73*	439
18	256	16	128	4201	123.69*	440
19	256	16	128	4134	113.32*	441
20	256	16	128	3845	116.21*	441

For problems 11-20, the final objective value from LOQO computations is the best objective value obtained during time and iteration limit. CPU times from (RDP) are within 450 seconds for all 20 problems. Since LOQO optimal values for problems 11-20 are best bound until time limit, the percentage of (RDP)/(P) increases. We suspect that if the problems instances 11-20 are solved optimally, the ratio (RDP)/(P) shows consistent upper bounds as instances 1-10. In the case of problems 11-20, the values in the column denoted as (RDP)/(P) marked with an asterisk (*) represent the estimated ratio of (RDP)/(P). Here, (P) values correspond to the final objective values obtained from the LOQO optimizer at the conclusion of the iteration limit.

Note that (RDP) achieves tight bound to (P) with fast CPU times. Also, for each product, 40 discrete price points are practically realistic bounds for price determination for retailers. Therefore, (RDP) can provide retailers valuable tool to determine optimal pricing guidelines in mixed logit environment.

5. **Conclusions.** In this paper, we developed a Lagrangian dual formulation and fast solution algorithm for a relaxation of the finite mixed logit profit maximization problem. The proposed relaxation is obtained by disaggregating and discretizing the price variables into segment-dependent discrete values and the resulting relaxation problem is a binary fractional programming problem. We solved the Lagrangian dual problem using a deflected subgradient minimization algorithm. In computational results, we compared nonlinear programming formulation with its approximation near particular points as well as in general price vectors. We also compared the objective values of the (RDP) with the nonlinear programming bound. As noted in the paper, solving the nonlinear formulation of mixed logit profit maximization problem is a hard challenge. We observed solver failure due to memory exhaustion in commercial solver AMPL LOQO in several occasions. For large scale problem instances, our relaxation bound and its solution are a valuable tool to compute optimal price vectors. Even though recent progress in the unconstrained

maximization of the mixed logit problem, when linear constraints are added to the formulation, its Lagrangian subproblem contains difficult subproblems. For constrained mixed logit problem, further research for the exact algorithms is required.

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