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ESSENTIAL HYPERIDEAL AND ESSENTIAL FUZZY HYPERIDEALS IN HYPERSEMIGROUPS

Nutthanicha Kaewmanee and Thiti Gaketem*

Department of Mathematics
School of Science
University of Phayao
Maeka, Muang, Phayao 56000, Thailand
64202149@up.ac.th; *Corresponding author: thiti.ga@up.ac.th

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Abstract. *Hyperstructure theory was first studied in 1934 by Marty. This they are widely studied from theoretical point of view and for their pure and applied mathematics. Essential ideal in studied by Medhi et al. Later in 2020, Baupradist et al. studied essential ideals and essential fuzzy ideals in semigroups. In this paper, we give the concepts of essential hyperideals and essential fuzzy hyperideals in hypersemigroups. We investigate the properties of essential hyperideals and essential fuzzy hyperideals in semigroups. Moreover, we show some relationships between essential hyperideals and essential fuzzy hyperideals.*

Keywords: Essential hyperideals, Essential fuzzy hyperideals, Minimal essential hyperideals, Minimal essential fuzzy hyperideals

1. **Introduction.** Hyperstructure theory was first studied in 1934 by Marty [1], which it is a generalization of ordinary algebraic structures. This they are widely studied from theoretical point of view and for their applications to many subjects of pure and applied properties. In 1965, Zadeh [2] gave the concept of a fuzzy set. The theory of fuzzy semigroups has been first considered by Kuroki [3]. The concept of fuzzy hyperalgebraic theory developed from fuzzy set in 2000 by Davvaz [4]. The essential ideals of rings have been studied in 2008 by Medhi et al. [5]. In 2013, Medhi and Saikia [6] discussed the concept of T-fuzzy essential ideals and studied the properties of T-fuzzy essential ideals. In 2017, Wani and Pawar [7] studied the concept of essential ideals ternary semirings. In 2020, Baupradist et al. [8] studied properties of the essential ideals and essential fuzzy ideals in semigroups, together with 0-essential ideals and 0-essential fuzzy ideals in semigroups. In 2021, Chinram and Gaketem [9] investigated essential (m,n)-ideals and essential fuzzy (m,n)-ideals in semigroups. Moreover, Gaketem et al. [10, 11] used knowledge of essential ideals in semigroups to study essential ideals in UP-algebra. In 2023, Panpetch et al. [12] studied essential bi-ideals and fuzzy essential bi-ideals in semigroups. In the same year, Khamrot and Gaketem [13] studied essential ideals in bipolar fuzzy set. Recently, Rittichuai et al. [14] studied essential ideals and essential fuzzy ideals in ternary semigroups.

The aim of this paper is to extend essential ideals and essential fuzzy ideals to hyperideals in hypersemigroups in which the main results are divided in the following sections. In Section 2, we shall extend concepts of essential ideals and essential fuzzy ideals to hyperideals in hypersemigroups. In Section 3, we study essential hyperideals and essential fuzzy hyperideals of hypersemigroups. In Section 4, we study 0-essential hyperideals and 0-essential fuzzy hyperideals of hypersemigroups with zero. Finally, we conclude the paper and give future work.

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2. **Preliminaries.** In this section, we review the definition of hypersemigroups and fuzzy hypersemigroups.

Let $\mathcal{H}$ be a non-empty set. A funtion $* : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$ is said to be *hyperoperation* on set $\mathcal{H}$, where $\mathcal{P}^*(\mathcal{H}) = \mathcal{P}(\mathcal{H})\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of $\mathcal{H}$.

A hyperoperation $(\mathcal{H}, *)$ is said to be *hypersemigroup* if $(a * b) * c = a * (b * c)$ for all $a, b, c \in \mathcal{H}$.

For $x \in \mathcal{H}$ and $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a set $\mathcal{H}$ defined as

$$\mathcal{A} * \mathcal{B} = \bigcup_{a\in\mathcal{A}, b\in\mathcal{B}} a * b,\ \mathcal{A} * x = \mathcal{A} * \{x\},\ \text{and } x * \mathcal{B} = \{x\} * \mathcal{B}.$$

A non-empty subset $\mathcal{B}$ of a hypersemigroup $\mathcal{H}$ is called a *subhypersemigroup* of $\mathcal{H}$ if $\mathcal{B}^2 \subseteq \mathcal{B}$.

A non-empty subset $\mathcal{B}$ of a hypersemigroup $\mathcal{H}$ is called a *left (right) hyperideal* of $\mathcal{H}$ if $\mathcal{H}\mathcal{B} \subseteq \mathcal{B}$ ($\mathcal{B}\mathcal{H} \subseteq \mathcal{B}$).

A *hyperideal* $\mathcal{B}$ of $\mathcal{H}$ is a non-empty subset which is both a left hyperideal and a right hyperideal of $\mathcal{H}$.

For any $a, b \in [0, 1]$, we have

$$a \vee b = \max\{a, b\} \text{ and } a \wedge b = \min\{a, b\}.$$

A *fuzzy set* φ of a non-empty set $\mathcal{T}$ is a function from $\mathcal{T}$ into unit closed interval $[0, 1]$ of real numbers, i.e., $\varphi : \mathcal{T} \to [0, 1]$.

For any two fuzzy sets of φ and τ of a non-empty set $\mathcal{T}$, we define

1) the *support* of φ instead of $\text{supp}(\varphi) = \{k \in \mathcal{T} | \varphi(k) \neq 0\}$,
2) $\varphi \subseteq \tau$ if $\varphi(k) \leq \tau(k)$ for all $k \in \mathcal{T}$,
3) $(\varphi \vee \tau)(k) = \varphi(k) \vee \tau(u) = \max\{\varphi(k), \tau(k)\}$ for each $k \in \mathcal{T}$,
4) $(\varphi \wedge \tau)(k) = \varphi(k) \wedge \tau(u) = \min\{\varphi(k), \tau(k)\}$ for each $k \in \mathcal{T}$.

For two fuzzy sets φ and τ in a hypersemigroup $\mathcal{H}$, define the product $\varphi \circ \tau$ as follows: for all $k \in \mathcal{H}$,

$$(\varphi \circ \tau)(k) = \begin{cases} \bigvee\limits_{(r,b)\in F_k} \{\{\varphi(r) \wedge \tau(b)\} | (r, b) \in F_k\} & \text{if } F_k \neq \emptyset, \\ 0 & \text{if } F_k = \emptyset, \end{cases}$$

where $F_k := \{(r, b) \in \mathcal{H} \times \mathcal{H} | k = rb\}$.

A *fuzzy hypersubsemigroup* φ of a hypersemigroup $\mathcal{H}$ if $\inf_{p\in k*r} \varphi(p) \geq \varphi(k) \wedge \varphi(r)$ for all $k, r \in \mathcal{H}$.

A *fuzzy hyperleft (hyperright) ideal* φ of a hypersemigroup $\mathcal{H}$ if $\inf_{p\in k*r} \varphi(p) \geq \varphi(r)$ ($\inf_{p\in k*r} \varphi(p) \geq \varphi(k)$) for all $k, r \in \mathcal{H}$. A *fuzzy hyperideal* φ of hypersemigroup $\mathcal{H}$ if it is both hyperleft ideal and hyperright ideal of $\mathcal{H}$ [4].

The *characteristic fuzzy set* $\chi_{\mathcal{B}}$ of a non-empty set $\mathcal{B}$ is defined as follows:

$$\chi_{\mathcal{B}} : T \to [0, 1], k \mapsto \begin{cases} 1 & \text{if } k \in \mathcal{B}, \\ 0 & \text{if } k \notin \mathcal{B}. \end{cases}$$

The following theorems are true.

Theorem 2.1. [4] *Let $\mathcal{B}$ be non-empty subset of hypersemigroup $\mathcal{H}$. Then $\mathcal{B}$ is a hyperideal of $\mathcal{H}$ if and only if characteristic function $\chi_{\mathcal{B}}$ is a hyperideal of $\mathcal{H}$.*

Theorem 2.2. [4] *Let $\mathcal{B}$ and $\mathcal{C}$ be non-empty subsets of a hypersemigroup $\mathcal{H}$. Then $\chi_{\mathcal{B}\cap\mathcal{C}} = \chi_{\mathcal{B}} \cap \chi_{\mathcal{C}}$ and $\chi_{\mathcal{B}} \circ \chi_{\mathcal{C}} = \chi_{\mathcal{B}\mathcal{C}}$.*

Theorem 2.3. *Let φ be a nonzero fuzzy set of a hypersemigroup $\mathcal{H}$. Then φ is a fuzzy hyperideal of $\mathcal{H}$ if and only if* $\text{supp}(\varphi)$ *is a hyperideal of $\mathcal{H}$.*

Proof: Assume that φ is a fuzzy hyperideal of $\mathcal{H}$ and let $k * r \in \text{supp}(\varphi)\mathcal{H}\,\text{supp}(\varphi)$. By assumption, we have

$$\inf_{p \in k*r} \varphi(p) \geq \varphi(k) \vee \varphi(r).$$

Therefore, $\inf_{p \in k*r} \varphi(p) \neq 0$. Thus, $k * r \in \text{supp}(\varphi)$. Hence, $\text{supp}(\varphi)$ is a hyperideal of $\mathcal{H}$. Conversely, assume that $\text{supp}(\varphi)$ is a hyperideal of $\mathcal{H}$ and let $k, r \in \mathcal{H}$. If $k, r \in \text{supp}(\varphi)$, then $k * r \in \text{supp}(\varphi)$. Thus, $\inf_{p \in k*r} \varphi(p) \geq \varphi(k) \vee \varphi(r)$. If $k, r \notin \text{supp}(\varphi)$, then $\inf_{p \in k*r} \varphi(p) \geq \varphi(k) \vee \varphi(r)$. Hence, φ is a fuzzy hyperideal of $\mathcal{H}$. □

3. **Essential Hyperideals and Essential Fuzzy Hyperideals in Hypersemigroups.** In this section, we will study concepts of essential hyperideals and fuzzy essential hyperideals in a hypersemigroup and properties of those.

Definition 3.1. *An* essential hyperideal *$\mathcal{B}$ of a hypersemigroup $\mathcal{H}$ if $\mathcal{B}$ is a hyperideal of $\mathcal{H}$ and $\mathcal{B} \cap \mathcal{C} \neq \emptyset$ for every hyperideal $\mathcal{C}$ of $\mathcal{H}$.*

Example 3.1. *(1) We have that a hypersemigroup $\mathcal{H}$ is an essential hyperideal of $\mathcal{H}$.*

(2) Let $\mathcal{H}$ be a hypersemigroup with zero. Then every hyperideal of $\mathcal{H}$ is an essential hyperideal of $\mathcal{H}$.

Theorem 3.1. *Let $\mathcal{B}$ be an essential hyperideal of a hypersemigroup $\mathcal{H}$. Then every hyperideal $\mathcal{B}_1$ of $\mathcal{H}$ containing $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$.*

Proof: Suppose that $\mathcal{B}_1$ is a hyperideal of $\mathcal{H}$ such that $\mathcal{B} \subseteq \mathcal{B}_1$ and let $\mathcal{C}$ be any hyperideal of $\mathcal{H}$. By assumption, $\mathcal{B} \cap \mathcal{C} \neq \emptyset$. Thus, $\emptyset \neq \mathcal{B} \cap \mathcal{C} \subseteq \mathcal{B}_1 \cap \mathcal{C}$. Hence, $\mathcal{B}_1 \cap \mathcal{C} \neq \emptyset$. Therefore, $\mathcal{B}_1$ is an essential hyperideal of $\mathcal{H}$. □

Theorem 3.2. *Let $\mathcal{B}$ and $\mathcal{C}$ be essential hyperideals of a hypersemigroup $\mathcal{H}$. Then $\mathcal{B} \cup \mathcal{C}$ and $\mathcal{B} \cap \mathcal{C}$ are essential hyperideals of $\mathcal{H}$.*

Proof: Since $\mathcal{B} \subseteq \mathcal{B} \cup \mathcal{C}$ and $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$, we have $\mathcal{B} \cup \mathcal{C}$ is an essential hyperideal of $\mathcal{H}$ by Theorem 3.1.

Since $\mathcal{B}$ and $\mathcal{C}$ are essential hyperideals of $\mathcal{H}$, we have $\mathcal{B}$ and $\mathcal{C}$ are hyperideals of $\mathcal{H}$. Thus, $\mathcal{B} \cap \mathcal{C}$ is a hyperideal of $\mathcal{H}$. Let $\mathcal{K}$ be a hyperideal of $\mathcal{H}$. Then $\mathcal{B} \cap \mathcal{K} \neq \emptyset$. Thus, there exists $u \in \mathcal{B} \cap \mathcal{K}$. Let $v \in \mathcal{C}$. Then $uv \in (\mathcal{B} \cap \mathcal{C}) \cap \mathcal{K}$. Thus, $(\mathcal{B} \cap \mathcal{C}) \cap \mathcal{K} \neq \emptyset$. Hence, $\mathcal{B} \cap \mathcal{C}$ is an essential hyperideal of $\mathcal{H}$. □

Theorem 3.3. *Let $\{\mathcal{K}_i | i \in \mathcal{A}\}$ be a non-empty collection of essential hyperideals of a semigroup S. Then $\cap_{i \in \mathcal{A}} \mathcal{K}_i$ is an essential hyperideal of $\mathcal{H}$.*

Proof: By assumption, we have $\{\mathcal{K}_i | i \in \mathcal{A}\}$ is a hyperideal of $\mathcal{H}$. Let $\mathcal{C}$ be any hyperideal of $\mathcal{H}$. Then $\cap_{i \in \mathcal{A}} \mathcal{K}_i \cap \mathcal{C}$ is a hyperideal of $\mathcal{H}$ and $\cap_{i \in \mathcal{A}} \mathcal{K}_i \cap \mathcal{C} \neq \emptyset$. Thus, $\cap_{i \in \mathcal{A}} \mathcal{K}_i$ is an essential hyperideal of $\mathcal{H}$. □

Definition 3.2. *A nonzero fuzzy hyperideal φ of a hypersemigroup $\mathcal{H}$ is called an* essential fuzzy hyperideal *of $\mathcal{H}$ if $\varphi \wedge \tau \neq 0$ for every nonzero fuzzy hyperideal τ of $\mathcal{H}$.*

Theorem 3.4. *Let φ be an essential fuzzy hyperideal of a hypersemigroup $\mathcal{H}$. If φ_1 is a fuzzy hyperideal of $\mathcal{H}$ such that $\varphi \subseteq \varphi_1$, then φ_1 is also an essential fuzzy hyperideal of $\mathcal{H}$.*

Proof: Let φ_1 be a fuzzy hyperideal of $\mathcal{H}$ such that $\varphi \subseteq \varphi_1$ and let τ be any fuzzy hyperideal of $\mathcal{H}$. Then, $\varphi \wedge \tau \neq 0$. Thus, $0 \neq \varphi \wedge \tau \subseteq \varphi_1 \wedge \tau$. So $\varphi_1 \wedge \tau \neq 0$. Hence, φ_1 is an essential fuzzy hyperideal of $\mathcal{H}$. □

Theorem 3.5. *Let φ_1 and φ_2 be essential fuzzy hyperideals of a hypersemigroup $\mathcal{H}$. Then $\varphi_1 \vee \varphi_2$ and $\varphi_1 \wedge \varphi_2$ are essential fuzzy hyperideals of $\mathcal{H}$.*

Proof: By Theorem 3.4, $\varphi_1 \vee \varphi_2$ is an essential fuzzy hyperideal of $\mathcal{H}$. Since φ_1 and φ_2 are essential fuzzy hyperideals of $\mathcal{H}$, we have $\varphi_1 \wedge \varphi_2$ is a fuzzy hyperideal of $\mathcal{H}$. Let τ be a nonzero fuzzy hyperideal of $\mathcal{H}$. Then, $\varphi_1 \wedge \tau \neq 0$. Thus, there exists $u \in \mathcal{H}$ such that

$$\varphi_1(u) \neq 0 \text{ and } \tau(u) \neq 0.$$

Since $\varphi_2 \neq 0$, let $v \in \mathcal{H}$ we have $\varphi_2(v) \neq 0$. Since φ_1 and φ_2 are fuzzy hyperideals of $\mathcal{H}$, we have

$$\inf_{p \in u*v} \varphi_1(p) \geq \varphi_1(u) \vee \varphi_1(v) \geq \varphi_1(u) > 0$$

and

$$\inf_{p \in u*v} \varphi_2(p) \geq \varphi_2(u) \vee \varphi_2(v) \geq \varphi_2(u) > 0.$$

So $\inf_{p \in u*v}(\varphi_1 \wedge \varphi_2)(p) = \varphi_1(uv) \wedge \varphi_2(uv) \neq 0$. Since τ is a fuzzy hyperideal of $\mathcal{H}$ and $\tau(u) \neq 0$, $\tau(uv) \neq 0$. Thus, $[(\varphi_1 \wedge \varphi_2) \wedge \tau](uv) \neq 0$. Hence, $[(\varphi_1 \wedge \varphi_2) \wedge \tau] \neq 0$. Therefore, $\varphi_1 \wedge \varphi_2$ is an essential fuzzy hyperideal of $\mathcal{H}$. □

Theorem 3.6. *Let $\{\varphi_i | i \in \mathcal{A}\}$ be a non-empty collection of essential fuzzy hyperideals of a semigroup $\mathcal{H}$. Then $\wedge_{i \in \mathcal{A}} \varphi_i$ is an essential fuzzy hyperideal of $\mathcal{H}$.*

Proof: By assumption, we have $\{\varphi_i | i \in \mathcal{A}\}$ is a fuzzy hyperideal of $\mathcal{H}$. Let τ be any fuzzy hyperideal of $\mathcal{H}$. Then $\wedge_{i \in \mathcal{A}} \varphi_i \wedge \tau$ is a fuzzy hyperideal of $\mathcal{H}$ and $\wedge_{i \in \mathcal{A}} \varphi_i \wedge \tau \neq 0$. Thus, $\wedge_{i \in \mathcal{A}} \varphi_i$ is an essential fuzzy hyperideal of $\mathcal{H}$. □

Theorem 3.7. *Let $\mathcal{B}$ be a hyperideal of a hypersemigroup $\mathcal{H}$. Then $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$ if and only if $\chi_{\mathcal{B}}$ is an essential fuzzy hyperideal of $\mathcal{H}$.*

Proof: Suppose that $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$ and let τ be any nonzero fuzzy hyperideal of $\mathcal{H}$. Then $\mathrm{supp}(\tau)$ is a hyperideal of $\mathcal{H}$. By assumption, we have $\mathcal{B}$ is a hyperideal of $\mathcal{H}$. Then $\chi_{\mathcal{B}}$ is a hyperideal of $\mathcal{H}$. Thus, $\mathcal{B} \cap \mathrm{supp}(\tau) \neq \emptyset$. So there exists $z \in \mathcal{B} \cap \mathrm{supp}(\tau)$. It implies that $z \in \mathcal{B}$ and $z \in \mathrm{supp}(\tau)$. Thus, $(\chi_{\mathcal{B}} \wedge \tau)(z) \neq 0$. Hence, $\chi_{\mathcal{B}} \wedge \tau \neq 0$. Therefore, $\chi_{\mathcal{B}}$ is an essential fuzzy hyperideal of $\mathcal{H}$.

Conversely, assume that $\chi_{\mathcal{B}}$ is an essential fuzzy hyperideal of $\mathcal{H}$ and let $\mathcal{C}$ be any hyperideal of $\mathcal{H}$. Then $\chi_{\mathcal{C}}$ is a nonzero fuzzy hyperideal of $\mathcal{H}$. Since $\chi_{\mathcal{B}}$ is an essential fuzzy hyperideal of $\mathcal{H}$, we have $\chi_{\mathcal{B}}$ is a fuzzy hyperideal of $\mathcal{H}$. Thus, $\chi_{\mathcal{B}} \wedge \chi_{\mathcal{C}} \neq 0$. So by Theorem 2.2, $\chi_{\mathcal{B} \cap \mathcal{C}} \neq 0$. Hence, $\mathcal{B} \cap \mathcal{C} \neq \emptyset$. Therefore, $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$. □

Theorem 3.8. *Let φ be a nonzero fuzzy hyperideal of a hypersemigroup $\mathcal{H}$. Then φ is an essential fuzzy hyperideal of $\mathcal{H}$ if and only if* $\mathrm{supp}(\varphi)$ *is an essential hyperideal of $\mathcal{H}$.*

Proof: Assume that φ is an essential fuzzy hyperideal of $\mathcal{H}$. Then $\mathrm{supp}(\varphi)$ is a hyperideal of $\mathcal{H}$. Let $\mathcal{B}$ be any hyperideal of $\mathcal{H}$. Then by Theorem 2.1, $\chi_{\mathcal{B}}$ is a hyperideal of $\mathcal{H}$. Since φ is an essential fuzzy hyperideal of $\mathcal{H}$, we have φ is a fuzzy hyperideal of $\mathcal{H}$. Thus, $\varphi \wedge \chi_{\mathcal{B}} \neq 0$. So there exists $z \in \mathcal{H}$ such that $(\varphi \wedge \chi_{\mathcal{B}})(z) \neq 0$. It implies that $\varphi(z) \neq 0$ and $\chi_{\mathcal{B}}(z) \neq 0$. Hence, $z \in \mathrm{supp}(\varphi) \cap \mathcal{B}$, so $\mathrm{supp}(\varphi) \cap \mathcal{B} \neq \emptyset$. It implies that $\mathrm{supp}(\varphi)$ is an essential hyperideal of $\mathcal{H}$.

Conversely, assume that $\mathrm{supp}(\varphi)$ is an essential hyperideal of $\mathcal{H}$ and let τ be a nonzero fuzzy hyperideal of $\mathcal{H}$. Then by Theorem 2.3, $\mathrm{supp}(\tau)$ is a hyperideal of $\mathcal{H}$. By assumption, $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\tau) \neq \emptyset$. So there exists $z \in \mathrm{supp}(\varphi) \cap \mathrm{supp}(\tau)$. This implies that $\varphi(z) \neq 0$ and $\tau(z) \neq 0$. Hence, $(\varphi \wedge \tau)(z) \neq 0$. Therefore, $\varphi \wedge \tau \neq 0$. We conclude that φ is an essential fuzzy hyperideal of $\mathcal{H}$. □

Next, we will focus on minimality of essential hyperideals and essential fuzzy hyperideals.

Definition 3.3. *An essential hyperideal $\mathcal{B}$ of a hypersemigroup $\mathcal{H}$ is called a* minimal essential hyperideal *if for every essential hyperideal $\mathcal{C}$ of $\mathcal{H}$ such that $\mathcal{C} \subseteq \mathcal{B}$, we have $\mathcal{B} = \mathcal{C}$.*

Definition 3.4. *An essential fuzzy hyperideal φ of a hypersemigroup $\mathcal{H}$ is called a* minimal fuzzy hyperideal *if for every essential fuzzy hyperideal τ of $\mathcal{H}$ such that $\tau \subseteq \varphi$, we have* $\mathrm{supp}(\varphi) = \mathrm{supp}(\tau)$.

Theorem 3.9. *Let $\mathcal{B}$ be a non-empty subset of a hypersemigroup $\mathcal{H}$. Then $\mathcal{K}$ is a minimal essential hyperideal of $\mathcal{H}$ if and only if $\chi_{\mathcal{K}}$ is a minimal essential fuzzy hyperideal of $\mathcal{H}$.*

Proof: Let $\mathcal{K}$ be a minimal essential hyperideal of $\mathcal{H}$. Then $\mathcal{K}$ is an essential hyperideal of $\mathcal{K}$. Thus, by Theorem 3.7, $\chi_{\mathcal{K}}$ is an essential fuzzy hyperideal of $\mathcal{H}$. Let φ be an essential fuzzy hyperideal of $\mathcal{H}$ such that $\varphi \subseteq \chi_{\mathcal{K}}$. Then $\mathrm{supp}(\varphi) \subseteq \mathrm{supp}(\chi_{\mathcal{K}}) = \mathcal{K}$. Thus, by Theorem 3.8, $\mathrm{supp}(\varphi)$ is an essential hyperideal of $\mathcal{H}$. Since $\mathcal{K}$ is a minimal essential hyperideal of $\mathcal{H}$, we have $\mathrm{supp}(\varphi) = \mathcal{K} = \mathrm{supp}(\chi_{\mathcal{K}})$. Thus, $\chi_{\mathcal{K}}$ is a minimal essential fuzzy hyperideal of $\mathcal{H}$.

Conversely, assume that $\chi_{\mathcal{K}}$ is a minimal essential fuzzy hyperideal of $\mathcal{S}$ and let $\mathcal{J}$ be an essential fuzzy hyperideal of $\mathcal{H}$ such that $\mathcal{J} \subseteq \mathcal{K}$. Then $\chi_{\mathcal{J}}$ is an essential fuzzy hyperideal of $\mathcal{H}$ such that $\chi_{\mathcal{J}} \subseteq \chi_{\mathcal{K}}$. Thus, $\mathcal{J} = \mathrm{supp}(\chi_{\mathcal{J}}) = \mathrm{supp}(\chi_{\mathcal{K}}) = \mathcal{K}$. Hence, $\mathcal{K}$ is a minimal essential hyperideal of $\mathcal{H}$. □

Next, we study relationship between prime (semiprime) essential hyperideals and prime (semiprime) essential fuzzy hyperideals in semigroups.

Definition 3.5. *Let $\mathcal{B}$ be an essential hyperideal of a hpersemigroup $\mathcal{H}$. Then*

(1) *$\mathcal{B}$ is called a* prime *if $uv \in \mathcal{B}$ implies $u \in \mathcal{B}$ or $v \in \mathcal{B}$ for all $u, v \in \mathcal{H}$;*
(2) *$\mathcal{B}$ is called a* semiprime *if $u^2 \in \mathcal{B}$ implies $u \in \mathcal{B}$ for all $u \in \mathcal{H}$.*

Definition 3.6. *Let φ be an essential fuzzy hyperideal of a hypersemigroup $\mathcal{H}$. Then*

(1) *φ is called a* prime *if $\inf_{p\in u*v} \varphi(p) \leq \varphi(u) \vee \varphi(v)$ for all $u, v \in \mathcal{H}$;*
(2) *φ is called a* semiprime *if $\inf_{p\in u*u} \varphi(p) \leq \varphi(u)$ for all $u \in \mathcal{H}$.*

Theorem 3.10. *A non-empty subset $\mathcal{B}$ of a hypersemigroup $\mathcal{H}$ is a prime (semiprime) essential hyperideal of $\mathcal{H}$ if and only if $\chi_{\mathcal{B}}$ is a prime (semiprime) essential fuzzy hyperideal of $\mathcal{H}$.*

Proof: Suppose that $\mathcal{B}$ is a prime essential hyperideal of $\mathcal{H}$. Then $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$. Thus, by Theorem 3.7, $\chi_{\mathcal{B}}$ is a fuzzy essential hyperideal of $\mathcal{H}$. Let u and v be any two elements of $\mathcal{H}$. If $uv \in \mathcal{B}$, then $u \in \mathcal{B}$ or $v \in \mathcal{B}$. Thus, $\inf_{p\in u*v} \chi_{\mathcal{B}}(p) \leq \chi_{\mathcal{B}}(u) \vee \chi_{\mathcal{B}}(v)$. If $uv \notin \mathcal{B}$, then $\inf_{p\in u*v} \chi_{\mathcal{B}}(p) \leq \chi_{\mathcal{B}}(u) \vee \chi_{\mathcal{B}}(v)$. Therefore, $\chi_{\mathcal{B}}$ is a prime essential fuzzy hyperideal of $\mathcal{H}$.

Conversely, assume that $\chi_{\mathcal{B}}$ is a prime essential fuzzy hyperideal of $\mathcal{H}$. Then $\chi_{\mathcal{B}}$ is an essential fuzzy hyperideal of $\mathcal{H}$. Thus, by Theorem 3.7, $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$. Let u and v be any two elements of $\mathcal{H}$ such that $uv \in \mathcal{B}$. Then $\inf_{p\in u*v} \chi_{\mathcal{B}}(p) = 1$. By assumption, $\inf_{p\in u*v} \chi_{\mathcal{B}}(p) \leq \chi_{\mathcal{B}}(u) \vee \chi_{\mathcal{B}}(v)$. Thus, $\chi_{\mathcal{B}}(u) \vee \chi_{\mathcal{B}}(v) = 1$. Hence, $u \in \mathcal{B}$ or $v \in \mathcal{B}$. Therefore, $\mathcal{B}$ is a prime essential hyperideal of $\mathcal{H}$. □

4. **0-Essential Hyperideals and 0-Essential Fuzzy Hyperideals.** In this section, let $\mathcal{H}$ be a hypersemigroup with zero. We define the definition and study properties of 0-essential hyperideals and 0-essential fuzzy hyperideals as follows.

Definition 4.1. *A nonzero hyperideal $\mathcal{B}$ of a hypersemigroup with zero $\mathcal{H}$ is called a* 0-essential hyperideal *of $\mathcal{H}$ if $\mathcal{B} \cap \mathcal{C} \neq \{0\}$ for every nonzero hyperideal $\mathcal{C}$ of $\mathcal{H}$.*

Theorem 4.1. *Let $\mathcal{B}$ be a 0-essential hyperideal of a hypersemigroup with zero $\mathcal{H}$. Then every hyperideal $\mathcal{B}_1$ of $\mathcal{H}$ containing $\mathcal{B}$ is a 0-essential hyperideal of $\mathcal{H}$.*

Proof: Let $\mathcal{B}$ be a 0-essential hyperideal of $\mathcal{H}$ and $\mathcal{B}_1$ be a hyperideal of $\mathcal{H}$ such that $\mathcal{B} \subseteq \mathcal{B}_1$. Let $\mathcal{C}$ be any nonzero hyperideal of $\mathcal{H}$. Then, $\mathcal{B} \cap \mathcal{C} \neq \{0\}$. Thus, $\{0\} \neq \mathcal{B} \cap \mathcal{C} \subseteq \mathcal{B}_1 \cap \mathcal{C}$. So, $\mathcal{B}_1 \cap \mathcal{C} \neq \{0\}$. Hence, $\mathcal{B}_1$ is a 0-essential hyperideal of $\mathcal{H}$. □

Theorem 4.2. *Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are 0-essential hyperideals of hypersemigroup with zero $\mathcal{H}$. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2$ are 0-essential hyperideals of $\mathcal{H}$.*

Proof: By Theorem 4.1, $\mathcal{B}_1 \cup \mathcal{B}_2$ is a 0-essential hyperideal of $\mathcal{H}$. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are 0-essential hyperideals of $\mathcal{H}$, we have $\mathcal{B}_1$ and $\mathcal{B}_2$ are hyperideals of $\mathcal{H}$. Thus, $\mathcal{B}_1 \cap \mathcal{B}_2$ is a hyperideal of $\mathcal{H}$. Let $\mathcal{C}$ be any nonzero hyperideal of $\mathcal{H}$. Then $\mathcal{B}_1 \cap \mathcal{C} \neq \{0\}$. Thus, there exists a nonzero $u \in \mathcal{B}_1 \cap \mathcal{C}$. Let $(u)_i$ be a hyperideal of $\mathcal{H}$ generated by u. Then $(u)_i \neq 0$.

Thus, there exists a nonzero element $v \in (u)_i \cap \mathcal{B}_2$. So, $y \in (\mathcal{B}_1 \cap \mathcal{B}_2) \cap \mathcal{C}$. Hence, $\mathcal{B}_1 \cap \mathcal{B}_2$ is a 0-essential hyperideal of $\mathcal{H}$. □

Definition 4.2. *A fuzzy hyperideal φ of hypersemigroup with zero $\mathcal{H}$ is called a* 0-essential fuzzy hyperideal *of $\mathcal{H}$ if* $\text{supp}(\varphi \wedge \tau) \neq \{0\}$ *for every fuzzy hyperideal τ of $\mathcal{H}$.*

Theorem 4.3. *Let φ be a 0-essential fuzzy hyperideal of a hypersemigroup with zero $\mathcal{H}$. If φ_1 is a fuzzy hyperideal of $\mathcal{H}$ such that $\varphi \subseteq \varphi_1$, then φ_1 is also a 0-essential fuzzy hyperideal of $\mathcal{H}$.*

Proof: Assume that φ is a 0-essential fuzzy hyperideal of $\mathcal{H}$ and φ_1 is a fuzzy hyperideal of $\mathcal{H}$ such that $\varphi \subseteq \varphi_1$. Let τ be any fuzzy hyperideal of $\mathcal{H}$. Then $\text{supp}(\varphi \wedge \tau) \neq \{0\}$. Thus, $\{0\} \neq \text{supp}(\varphi \wedge \tau) \subseteq \text{supp}(\varphi_1 \wedge \tau) \neq \{0\}$. So, $\text{supp}(\varphi_1 \wedge \tau) \neq \{0\}$. Hence, φ_1 is a 0-essential fuzzy hyperideal of $\mathcal{H}$. □

Theorem 4.4. *Assume that φ_1 and φ_2 are 0-essential fuzzy hyperideals of a hypersemigroup with zero $\mathcal{H}$. Then $\varphi_1 \vee \varphi_2$ and $\varphi_1 \wedge \varphi_2$ are 0-essential fuzzy hyperideals of $\mathcal{H}$.*

Proof: By Theorem 4.3, $\varphi_1 \vee \varphi_2$ is a 0-essential fuzzy hyperideal of $\mathcal{H}$. Since φ_1 and φ_2 are 0-essential fuzzy hyperideals of $\mathcal{H}$, we have f_1 and f_2 are fuzzy (m, n)-ideals of S. Thus, $\varphi_1 \wedge \varphi_2$ is a fuzzy hyperideal of $\mathcal{H}$. Let τ be any fuzzy hyperideal of $\mathcal{H}$. Then $\text{supp}(\varphi_1 \wedge \tau) \neq \{0\}$. Thus, there exists a nonzero element $u \in \mathcal{H}$ such that $(\varphi_1 \wedge \tau)(u) \neq 0$. Since φ_2 is a 0-essential fuzzy hyperideal of $\mathcal{H}$, we have $\text{supp}(\varphi_2)$ is a 0-essential fuzzy hyperideal of $\mathcal{H}$. Thus, $\text{supp}(\varphi_2) \wedge (u)_i \neq \{0\}$. So there exists $u \in \text{supp}(\varphi_2) \wedge (u)_i$. Hence, $\varphi_2(u) \neq 0$. Since φ_1 and τ are fuzzy hyperideals of $\mathcal{H}$, we have $((\varphi_1 \wedge \varphi_2) \wedge \tau)(u) \neq 0$. Thus, $\sup[(\varphi_1 \wedge \varphi_2) \wedge \tau] \neq \{0\}$. Hence, $\varphi_1 \wedge \varphi_2$ is a 0-essential fuzzy hyperideal of $\mathcal{H}$. □

Theorem 4.5. *A nonzero hyperideal $\mathcal{B}$ of a hypersemigroup with zero $\mathcal{H}$ is a 0-essential hyperideal if and only if $\chi_{\mathcal{B}}$ is a 0-essential fuzzy hyperideal of $\mathcal{H}$.*

Proof: Assume that $\mathcal{B}$ is a 0-essential hyperideal of $\mathcal{H}$ and let φ be a fuzzy hyperideal of $\mathcal{H}$. Then $\text{supp}(\varphi)$ is a nonzero hyperideal of $\mathcal{H}$. Thus, $\mathcal{B} \cap \text{supp}(\varphi) \neq \{0\}$. So there exists a nonzero element $u \in \mathcal{H}$ such that $u \in \mathcal{B} \cap \text{supp}(\varphi)$. Since $\mathcal{B}$ is a 0-essential hyperideal of $\mathcal{H}$, we have $\mathcal{B}$ is an essential hyperideal of $\mathcal{H}$. Thus, by Theorem 3.7, $\chi_{\mathcal{B}}$ is an essential hyperideal of $\mathcal{H}$. So, $(\chi_{\mathcal{B}} \wedge \varphi)(u) \neq 0$. Hence, $u \in \text{supp}(\chi_{\mathcal{B}} \wedge \varphi)$. Therefore, $\chi_{\mathcal{B}}$ is a 0-essential fuzzy hyperideal of $\mathcal{H}$.

Conversely, assume that $\chi_{\mathcal{B}}$ is a 0-essential fuzzy hyperideal of $\mathcal{H}$ and let $\mathcal{J}$ be a nonzero hyperideal of $\mathcal{H}$. Then $\chi_{\mathcal{J}}$ is a fuzzy hyperideal of $\mathcal{H}$. Thus, $\text{supp}(\chi_{\mathcal{B}} \cap \chi_{\mathcal{J}}) \neq \{0\}$ so by Theorem 2.2, $\chi_{\mathcal{B}} \wedge \chi_{\mathcal{J}} = \chi_{\mathcal{B} \cap \mathcal{J}} \neq \{0\}$. Hence, $\mathcal{B} \cap \mathcal{J} \neq \emptyset$. Therefore, $\mathcal{B}$ is a 0-essential hyperideal of $\mathcal{H}$. □

Theorem 4.6. *Let φ be a nonzero fuzzy hyperideal of a hypersemigroup with zero $\mathcal{H}$. Then φ is a 0-essential fuzzy hyperideal of $\mathcal{H}$ if and only if* $\text{supp}(\varphi)$ *is a 0-essential hyperideal of $\mathcal{H}$.*

Proof: Assume that φ is a 0-essential fuzzy hyperideal of $\mathcal{H}$ and let $\mathcal{J}$ be a nontrivial hyperideal of $\mathcal{H}$. Then by Theorem 2.1, $\chi_{\mathcal{J}}$ is a nonzero fuzzy hyperideal of $\mathcal{H}$. By assumption, φ is a nonzero hyperideal of $\mathcal{H}$. Thus, $\varphi \wedge \chi_{\mathcal{J}} \neq 0$. So there exists a nonzero element $u \in \mathcal{H}$ such that $(\varphi \wedge \chi_{\mathcal{J}})(u) \neq 0$. It implies that $\varphi(u) \neq 0$ and $\chi_{\mathcal{J}}(u) \neq 0$. Hence, $u \in \text{supp}(\varphi) \cap \mathcal{J}$ so $\text{supp}(\varphi) \cap \mathcal{J} \neq \{0\}$. Therefore, $\text{supp}(\varphi)$ is a 0-essential hyperideal of $\mathcal{H}$.

Conversely, assume that $\text{supp}(\varphi)$ is a 0-essential hyperideal of $\mathcal{H}$ and let τ be a nonzero hyperideal of $\mathcal{H}$. Then by Theorem 2.3, $\text{supp}(\tau)$ is a nontrivial zero hyperideal of $\mathcal{H}$. By assumption, $\text{supp}(\varphi)$ is a nonzero hyperideal of $\mathcal{H}$. Thus, $\text{supp}(\varphi) \cap \text{supp}(\tau) \neq 0$. So, there exists $u \in \text{supp}(\varphi) \cap \text{supp}(\tau)$, this implies that $\varphi(u) \neq 0$ and $\tau(u) \neq 0$ for all $u \in \mathcal{H}$. Hence, $(\varphi \wedge \tau)(u) \neq 0$ for all $u \in \mathcal{H}$. Therefore, $\varphi \wedge \tau \neq 0$. We conclude that φ is a 0-essential fuzzy hyperideal of $\mathcal{H}$. □

Next, we will focus on minimality of 0-essential hyperideals and 0-essential fuzzy hyperideals.

Definition 4.3. *A* 0*-essential hyperideal* $\mathcal{B}$ *of a hypersemigroup* $\mathcal{H}$ *is called a* minimal 0-essential hyperideal *if for every* 0*-essential hyperideal* $\mathcal{C}$ *of* $\mathcal{H}$ *such that* $\mathcal{C} \subseteq \mathcal{B}$*, we have* $\mathcal{B} = \mathcal{C}$*.*

Example 4.1. *Let* $(\mathbb{Z}_{12}, +)$ *be a hypersemigroup. Then* $\{0, 2, 4, 6, 8, 10\}$ *is a unique* 0*-essential hyperideal of* $\mathbb{Z}_{12}$*.*

Definition 4.4. *A* 0*-essential fuzzy hyperideal* φ *of a hypersemigroup* $\mathcal{H}$ *is called a* minimal fuzzy hyperideal *if for every* 0*-essential fuzzy hyperideal* τ *of* $\mathcal{H}$ *such that* $\tau \subseteq \varphi$*, we have* $\text{supp}(\varphi) = \text{supp}(\tau)$*.*

Theorem 4.7. *Let* $\mathcal{B}$ *be a non-empty subset of a hypersemigroup* $\mathcal{H}$*. Then* $\mathcal{K}$ *is a minimal* 0*-essential hyperideal of* $\mathcal{H}$ *if and only if* $\chi_{\mathcal{K}}$ *is a minimal* 0*-essential fuzzy hyperideal of* $\mathcal{H}$*.*

Proof: It follows from Theorem 3.9. □

Next, we prove relationship between prime (semiprime) essential hyperideals and prime (semiprime) essential fuzzy hyperideals in semigroups.

Definition 4.5. *Let* $\mathcal{B}$ *be a* 0*-essential hyperideal of a hypersemigroup* $\mathcal{H}$*. Then*

(1) $\mathcal{B}$ *is called a* prime *if* $uv \in \mathcal{B}$ *implies* $u \in \mathcal{B}$ *or* $v \in \mathcal{B}$ *for all* $u, v \in \mathcal{H}$*;*

(2) $\mathcal{B}$ *is called a* semiprime *if* $u^2 \in \mathcal{B}$ *implies* $u \in \mathcal{B}$ *for all* $u \in \mathcal{H}$*.*

Definition 4.6. *Let* φ *be a* 0*-essential fuzzy hyperideal of a hypersemigroup* $\mathcal{H}$*. Then*

(1) φ *is called a* prime *if* $\inf_{p \in u*v} \varphi(p) \leq \varphi(u) \vee \varphi(v)$ *for all* $u, v \in \mathcal{H}$*;*

(2) φ *is called a* semiprime *if* $\inf_{p \in u*u} \varphi(p) \leq \varphi(u)$ *for all* $u \in \mathcal{H}$*.*

Theorem 4.8. *A non-empty subset* $\mathcal{B}$ *of a hypersemigroup* $\mathcal{H}$ *is a prime (semiprime)* 0*-essential hyperideal of* $\mathcal{H}$ *if and only if* $\chi_{\mathcal{B}}$ *is a prime (semiprime)* 0*-essential fuzzy hyperideal of* $\mathcal{H}$*.*

Proof: It follows from Theorem 3.10. □

5. **Conclusion.** In Section 3, we define essential hyperideals and essential fuzzy hyperideals in semigroups. We present that the union and intersection of essential hyperideals (essential fuzzy hyperideals) of hypersemigroup $\mathcal{H}$ are also an essential hyperideal (essential fuzzy hyperideal) of $\mathcal{H}$. Moreover, we prove some relationship between essential hyperideals and essential fuzzy hyperideals. In Section 4, we define 0-essential hyperideals and 0-essential fuzzy hyperideals in semigroups with zero.

In the future work, we can discuss essential i-ideals and essential fuzzy i-ideals in n-ary semigroups and algebraic systems.

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REFERENCES

[1] F. Marty, Sur une generalization de la notion de group, *Proc. of the 8th Congress des Mathematician Scandinave*, pp.45-49, 1934.

[2] L. A. Zadeh, Fuzzy sets, *Information and Control*, vol.8, pp.338-353, 1965.

[3] N. Kuroki, Fuzzy bi-ideals in semigroup, *Comment. Math. Univ. St. Paul*, vol.5, pp.128-132, 1979.

[4] B. Davvaz, Fuzzy hyperideals in semihypergroups, *Italian Journal of Pure and Applied Mathematics*, vol.8, pp.67-74, 2000.

[5] U. Medhi, K. Rajkhowa, L. K. Barthakur and H. K. Saikia, On fuzzy essential ideals of ring, *Advances in Fuzzy Sets and Systems*, vol.3, pp.287-299, 2008.

[6] U. Medhi and H. K. Saikia, On T-fuzzy essential ideals of rings, *Journal of Pure and Applied Mathematics*, vol.89, pp.343-353, 2013.

[7] S. Wani and K. Pawar, On essential ideals of a ternary semiring, *Sohag Journal of Mathematics*, vol.4, no.3, pp.65-68, 2017.

[8] S. Baupradist, B. Chemat, K. Palanivel and R. Chinram, Essential ideals and essential fuzzy ideals in semigroups, *Journal of Discrete Mathematical Sciences and Cryptography*, vol.24, no.1, pp.223-233, 2020.

[9] R. Chinram and T. Gaketem, Essential (m, n)-ideal and essential fuzzy (m, n)-ideals in semigroups, *ICIC Express Letters*, vol.15, no.10, pp.1037-1044, 2021.

[10] T. Gaketem and A. Iampan, Essential UP-ideals and t-essential fuzzy UP-ideals of UP-algebra, *ICIC Express Letters*, vol.15, no.12, pp.1283-1289, 2021.

[11] T. Gaketem, P. Khamrot and A. Iampan, Essential UP-filters and t-essential fuzzy UP-filters of UP-algebra, *ICIC Express Letters*, vol.16, no.10, pp.1057-1062, 2022.

[12] N. Panpetch, T. Muangngao and T. Gaketem, Some essential fuzzy bi-ideals and essential fuzzy bi-ideals in a semigroup, *Journal of Mathematics and Computer Science*, vol.28, no.4, pp.326-334, 2023.

[13] P. Khamrot and T. Gaketem, Essential bipolar fuzzy ideals of UP-algebra, *International Journal of Analysis and Applications*, vol.21, pp.1-12, 2023.

[14] R. Rittichuai, A. Iampan, R. Chinram and P. Singavanda, Essential ideals and their fuzzifications of ternary semigroups, *ICIC Express Letters*, vol.17, no.2, pp.191-199, 2023.