ICIC Express Letters
Volume **19**, Number **4**, April **2025**
ICIC International ©2025 ISSN 1881-803X
pp. **403–410**

POWERS OF THE GENERATING MATRIX FOR THE GENERALIZED FIBONACCI SEQUENCE OF ORDER k

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Received April 2024; accepted July 2024

Abstract. *We establish that under certain conditions, the generating matrix for the generalized Fibonacci sequence of order k can be diagonalized by a Vandermonde matrix. The purpose of this work is to investigate the properties of the generating matrix for the generalized Fibonacci sequence of order k and to derive new results related to its diagonalizability and applications. The results include a closed-form expression for the matrix's powers and the determinant of a related Toeplitz matrix.*
Keywords: Fibonacci, Lucas, Matrices, Vandermonde, Toeplitz, Powers

1. **Introduction.** The generalized Fibonacci sequence $(T_n)_{n\geq 0}$ of order $k \geq 2$ is defined by $T_0 = T_1 = \cdots = T_{k-2} = 0$, $T_{k-1} = 1$, and for all $n \geq k$,

$$T_n = a_1T_{n-1} + a_2T_{n-2} + \cdots + a_kT_{n-k}, \tag{1}$$

where $a_1, \ldots, a_k$ are nonzero integers. For $k = 2$, the sequence $(T_n)_{n\geq 0}$ reduces to the usual Lucas sequence. The recurrence relation for the generalized Fibonacci sequence can be expressed in matrix form. Let

$$\mathbf{T}_n = (T_n \quad T_{n-1} \quad \cdots \quad T_{n-k+1})^T$$

and define the matrix $\mathbf{A}_k$ by

$$\mathbf{A}_k = \begin{pmatrix} a_1 & a_2 & \ldots & a_{k-1} & a_k \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

Then we have the equation $\mathbf{T}_n = \mathbf{A}_k\mathbf{T}_{n-1}$. The matrix $\mathbf{A}_k$ is said to be the generating (or companion) matrix of the generalized Fibonacci sequence of order k. For the cases $k = 2$ and $k = 3$, the mth ($m \geq 1$) power of the generating matrix $\mathbf{A}_k$ is extensively investigated (see, for example, Cerda-Morales [1], Shannon and Horadam [2], and Waddill [3]), yielding the forms shown in the following equations:

$$\mathbf{A}_2^m = \begin{pmatrix} T_{m+1} & a_2T_m \\ T_m & a_2T_{m-1} \end{pmatrix} \text{ and } \mathbf{A}_3^m = \begin{pmatrix} T_{m+2} & T_{m+3} - a_1T_{m+2} & a_3T_{m+1} \\ T_{m+1} & T_{m+2} - a_1T_{m+1} & a_3T_m \\ T_m & T_{m+1} - a_1T_m & a_3T_{m-1} \end{pmatrix}.$$

DOI: 10.24507/icicel.19.04.403

This study investigates the power of the generating matrix $\mathbf{A}_k$ for higher orders. We prove that, under specific conditions, the generating matrix of the generalized Fibonacci sequence of order k can be diagonalized by a Vandermonde matrix, enabling a closed-form expression for the matrix's power and recovery of well-known Lucas sequence identities. Prasad and Mahato [4] explored a specific form of the matrix with $a_1 = a_2 = \cdots = a_k = 1$, leading to cryptographic applications. Additionally, we present a closed form for the determinant of a Toeplitz matrix whose entries are the generalized Fibonacci sequence of order k. Toeplitz matrices, noted for their unique structure and applications in signal processing, control theory, and numerical analysis (Gray [5], Trench [6]), are complemented by our result, offering a general method for determining such matrices' determinants. The advantages of these methods include leveraging matrix diagonalization for closed-form expressions, despite the complexity of generalizing to higher-order sequences and intricate calculations.

The remainder of this paper is organized as follows. Section 2 covers definitions and preliminaries related to the generalized Fibonacci sequence, its generating matrix, and the Vandermonde matrix. Section 3 presents our main result on the diagonalizability of the generating matrix and derives its power's closed form, offering a combinatorial perspective different from Taher and Rachidi [7]. In Section 4, we link our results to Lucas sequences, recover known identities, and provide a closed form for the determinant of a Toeplitz matrix containing generalized Fibonacci sequences.

2. **Preliminaries.** Lemma 2.1 is a curious spectral property of the generating matrix $\mathbf{A}_k$. Lemma 2.2 provides an explicit formula for the generalized Fibonacci sequence of order k (see, for example, Kalman [8] and Levesque [9]).

Lemma 2.1. *Suppose that the zeros $\alpha_1, \ldots, \alpha_k$ of the characteristic polynomial $p(x) = x^k - a_1x^{k-1} - \cdots - a_k$ of (1) are distinct. Then $\alpha_1, \ldots, \alpha_k$ are the eigenvalues of the matrix $\mathbf{A}_k$. Moreover, the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ corresponding to $\alpha_1, \ldots, \alpha_k$, respectively, are given by*

$$\mathbf{v}_j = \begin{pmatrix} \alpha_j^{k-1} & \alpha_j^{k-2} & \cdots & 1 \end{pmatrix}^T, \quad 1 \leq j \leq k.$$

Proof: Let λ be an eigenvalue of the generating matrix $\mathbf{A}_k$. We require that

$$\det(\lambda \mathbf{I}_k - \mathbf{A}_k) = 0,$$

where $\mathbf{I}_k$ is the identity matrix of order k. We claim that the determinant on the left-hand side of the equation has the form

$$\det(\lambda \mathbf{I}_k - \mathbf{A}_k) = \lambda^k - a_1\lambda^{k-1} - \cdots - a_k.$$

This would show that λ is a zero of the characteristic polynomial $p(x)$, and thus proving the first part of the theorem. This claim follows from a general formula that, for any x, $b_1, \ldots, b_k$ $(k \geq 2)$, we have

$$\det(x\mathbf{I}_k - \mathbf{B}_k) = x^k - b_1x^{k-1} - \cdots - b_k,$$

where $\mathbf{B}_k$ is a square matrix of the form

$$\mathbf{B}_k = \begin{pmatrix} b_1 & b_2 & \ldots & b_{k-1} & b_k \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

We prove the formula by induction on k. For $k = 2$, we have

$$\det(x\mathbf{I}_2 - \mathbf{B}_2) = x(x - b_1) - (-b_2)(-1) = x^2 - b_1x - b_2.$$

Hence, the formula holds for $k = 2$. We assume that the formula holds for $k - 1$ where $k \geq 3$ and prove that it also holds for k. Expanding the determinant along the last column of the matrix $x\mathbf{I}_k - \mathbf{B}_k$, we obtain

$$\det(x\mathbf{I}_k - \mathbf{B}_k) = (-1)^{k+1}(-b_k)M_{1k} + (-1)^{k+k}xM_{kk},$$

where M_{ij} is the minor of the entry in row i and column j of the matrix. We observe that M_{1k} is the determinant of a $(k-1)\times(k-1)$ upper triangular matrix whose main diagonal entries are -1, so $M_{1k} = (-1)^{k-1}$. The minor M_{kk} is the determinant of the matrix of the form $x\mathbf{I}_{k-1} - \mathbf{B}_{k-1}$. Applying the induction hypothesis, we obtain

$$\begin{aligned}\det(x\mathbf{I}_k - \mathbf{B}_k) &= (-1)^{k+1}(-b_k)(-1)^{k-1} + (-1)^{k+k}x\left(x^{k-1} - b_1x^{k-2} - \cdots - b_{k-1}\right)\\ &= x^k - b_1x^{k-1} - \cdots - b_k,\end{aligned}$$

completing the proof of the formula by induction. To prove the second part of the theorem, it suffices to observe the identities

$$\mathbf{A}_k\mathbf{v}_j = \alpha_j\mathbf{v}_j$$

for $j = 1, \ldots, k$. These identities follow, since $\alpha_j^k = a_1\alpha_j^{k-1} + \cdots + a_k$ for $j = 1, \ldots, k$. □

Lemma 2.2. *Suppose that the zeros $\alpha_1, \ldots, \alpha_k$ of the characteristic polynomial $p(x)$ of (1) are distinct. Then the general term of the generalized Fibonacci sequence $(T_n)_{n\geq 0}$ of order k is given explicitly by*

$$T_n = \sum_{i=1}^{k} \frac{\alpha_i^n}{\prod_{\substack{j=1\\ j\neq i}}^{k}(\alpha_i - \alpha_j)}, \quad n \geq 0. \tag{2}$$

We now introduce the square matrix $\mathbf{V}_k$ of order k, known as the Vandermonde matrix, defined as follows:

$$\mathbf{V}_k = \begin{pmatrix} \alpha_1^{k-1} & \alpha_2^{k-1} & \ldots & \alpha_k^{k-1}\\ \alpha_1^{k-2} & \alpha_2^{k-2} & \ldots & \alpha_k^{k-2}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \ldots & 1 \end{pmatrix}, \tag{3}$$

where $\alpha_1, \ldots, \alpha_k$ are distinct. The multiplicative inverse of this matrix is of particular importance in our analysis, as it plays a pivotal role in the development of the forthcoming results.

Theorem 2.1. *The determinant of the Vandermonde matrix $\mathbf{V}_k$ is given by*

$$\det(\mathbf{V}_k) = \prod_{1\leq i<j\leq k}(\alpha_i - \alpha_j). \tag{4}$$

Proof: We use elementary column operations and the Laplace expansion. We start by subtracting consecutive columns and leaving alone the last column, which gives us

$$\det(\mathbf{V}_k) = \begin{vmatrix} \alpha_1^{k-1} - \alpha_2^{k-1} & \alpha_2^{k-1} - \alpha_3^{k-1} & \ldots & \alpha_k^{k-1}\\ \alpha_1^{k-2} - \alpha_2^{k-2} & \alpha_2^{k-2} - \alpha_3^{k-2} & \ldots & \alpha_k^{k-2}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_1 - \alpha_2 & \alpha_2 - \alpha_3 & \ldots & \alpha_k\\ 0 & 0 & \ldots & 1 \end{vmatrix}.$$

Next, we factor out the term $(\alpha_i - \alpha_{i+1})$ for $1 \le i \le k-1$ from each column, obtaining

$$\det(\mathbf{V}_k) = \prod_{i=1}^{k-1}(\alpha_i - \alpha_{i+1}) \begin{vmatrix} \sum_{\ell=0}^{k-2} \alpha_1^\ell \alpha_2^{k-2-\ell} & \sum_{\ell=0}^{k-2} \alpha_2^\ell \alpha_3^{k-2-\ell} & \dots & \alpha_k^{k-1} \\ \sum_{\ell=0}^{k-3} \alpha_1^\ell \alpha_2^{k-3-\ell} & \sum_{\ell=0}^{k-3} \alpha_2^\ell \alpha_3^{k-3-\ell} & \dots & \alpha_k^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & \alpha_k \\ 0 & 0 & \dots & 1 \end{vmatrix}.$$

We then compute the determinant using Laplace expansion along the last row:

$$\det(\mathbf{V}_k) = \prod_{i=1}^{k-1}(\alpha_i - \alpha_{i+1}) \begin{vmatrix} \sum_{\ell=0}^{k-2} \alpha_1^\ell \alpha_2^{k-2-\ell} & \sum_{\ell=0}^{k-2} \alpha_2^\ell \alpha_3^{k-2-\ell} & \dots & \sum_{\ell=0}^{k-2} \alpha_{k-1}^\ell \alpha_k^{k-2-\ell} \\ \sum_{\ell=0}^{k-3} \alpha_1^\ell \alpha_2^{k-3-\ell} & \sum_{\ell=0}^{k-3} \alpha_2^\ell \alpha_3^{k-3-\ell} & \dots & \sum_{\ell=0}^{k-3} \alpha_{k-1}^\ell \alpha_k^{k-3-\ell} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix}.$$

Repeating the same process to the above determinant and so on, we finally arrive at the identity (4) as desired. □

Theorem 2.2. *The multiplicative inverse of the Vandermonde matrix $\mathbf{V}_k$ is given by*

$$\mathbf{V}_k^{-1} = \left(\frac{\alpha_i^{j-1} - a_1\alpha_i^{j-2} - a_2\alpha_i^{j-3} - \cdots - a_{j-1}}{\prod_{\substack{n=1\\n\neq i}}^{k}(\alpha_i - \alpha_n)} \right)_{1\le i,j\le k}. \tag{5}$$

Proof: It suffices to verify that the product of the matrix on the right-hand side in the above equation and the Vandermonde matrix $\mathbf{V}_k$ gives the identity matrix $\mathbf{I}_k$ of order k. The entry in row i and column j of this product is given by

$$\begin{aligned} &\alpha_1^{k-i} \cdot \frac{\alpha_1^{j-1} - a_1\alpha_1^{j-2} - a_2\alpha_1^{j-3} - \cdots - a_{j-1}}{\prod_{\substack{n=1\\n\neq 1}}^{k}(\alpha_1 - \alpha_n)} + \cdots \\ &+ \alpha_k^{k-i} \cdot \frac{\alpha_k^{j-1} - a_1\alpha_k^{j-2} - a_2\alpha_k^{j-3} - \cdots - a_{j-1}}{\prod_{\substack{n=1\\n\neq k}}^{k}(\alpha_k - \alpha_n)} \\ &= \sum_{m=1}^{k} \frac{\alpha_m^{k-i+j-1}}{\prod_{\substack{n=1\\n\neq m}}^{k}(\alpha_m - \alpha_n)} - a_1 \sum_{m=1}^{k} \frac{\alpha_m^{k-i+j-2}}{\prod_{\substack{n=1\\n\neq m}}^{k}(\alpha_m - \alpha_n)} - \cdots - a_{j-1} \sum_{m=1}^{k} \frac{\alpha_m^{k-i}}{\prod_{\substack{n=1\\n\neq m}}^{k}(\alpha_m - \alpha_n)} \\ &= T_{k-i+j-1} - a_1 T_{k-i+j-2} - \cdots - a_{j-1} T_{k-i}, \end{aligned}$$

where the last equality follows from Lemma 2.2. We see that for $i = j$, this entry is

$$T_{k-1} - a_1 T_{k-2} - \cdots - a_{i-1} T_{k-i} = 1 - a_1(0) - \cdots - a_{i-1}(0) = 1$$

and for $i \neq j$, this entry is

$$T_{k-i+j-1} - a_1 T_{k-i+j-2} - \cdots - a_{j-1} T_{k-i} = 0 - a_1(0) - \cdots - a_{j-1}(0) = 0.$$

Hence, the product of these two matrices yields the identity matrix of order k, and therefore, the inverse of the Vandermonde matrix is justified. □

Remark 2.1. *The numerators of the entries of the matrix* (5) *satisfy the recurrence* $(v_{i,j})_{1\le i,j\le k}$ *given by* $v_{i,1} = 1$ *for* $1 \le i \le k$ *and* $v_{i,j} = \alpha_i v_{i,j-1} - a_{j-1}$ *for* $1 \le i, j \le k$.

3. **Main Theorem.** Now, we are ready to present our main theorem.

Theorem 3.1. *The mth ($m \geq 1$) power of the generating matrix of the generalized Fibonacci sequence of order k is given by*

$$\mathbf{A}_k^m = \left(T_{m+k-i+j-1} - \sum_{\ell=1}^{j-1} a_\ell T_{m+k-i+j-\ell-1} \right)_{1 \leq i,j \leq k}. \tag{6}$$

Proof: By Lemma 2.1, diagonalizing the generating matrix of generalized Fibonacci sequence of order k and raising to the mth power, we obtain

$$\mathbf{A}_k^m = \begin{pmatrix} \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_k^{k-1} \\ \alpha_1^{k-2} & \alpha_2^{k-2} & \dots & \alpha_k^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1^m & 0 & \dots & 0 \\ 0 & \alpha_2^m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_k^m \end{pmatrix} \begin{pmatrix} \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_k^{k-1} \\ \alpha_1^{k-2} & \alpha_2^{k-2} & \dots & \alpha_k^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \alpha_1^{m+k-1} & \alpha_2^{m+k-1} & \dots & \alpha_k^{m+k-1} \\ \alpha_1^{m+k-2} & \alpha_2^{m+k-2} & \dots & \alpha_k^{m+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^m & \alpha_2^m & \dots & \alpha_k^m \end{pmatrix} \begin{pmatrix} \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_k^{k-1} \\ \alpha_1^{k-2} & \alpha_2^{k-2} & \dots & \alpha_k^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}^{-1}.$$

By Theorem 2.2, substituting the multiplicative inverse of the Vandermonde matrix (5) into the above equation, we obtain

$$\mathbf{A}_k^m = \left(\alpha_j^m \mathbf{v}_j \right)_{1 \leq j \leq k} \left(\frac{\alpha_i^{j-1} - a_1 \alpha_i^{j-2} - a_2 \alpha_i^{j-3} - \dots - a_{j-1}}{\prod_{\substack{n=1 \\ n \neq i}}^{k} (\alpha_i - \alpha_n)} \right)_{1 \leq i,j \leq k},$$

where $\mathbf{v}_j$'s are defined as in the statement of Lemma 2.1. The entry in row i and column j of this product is given by

$$\sum_{i=1}^{k} \frac{\alpha_i^{m+k-i+j-1}}{\prod_{\substack{j=1 \\ j \neq i}}^{k} (\alpha_i - \alpha_j)} - \sum_{\ell=1}^{j-1} a_\ell \sum_{i=1}^{k} \frac{\alpha_i^{m+k-i+j-\ell-1}}{\prod_{\substack{j=1 \\ j \neq i}}^{k} (\alpha_i - \alpha_j)} = T_{m+k-i+j-1} - \sum_{\ell=1}^{j-1} a_\ell T_{m+k-i+j-\ell-1},$$

where the equality follows from Lemma 2.2. Therefore, the proof is complete. □

Another way to find the power of the generating matrix is by the reduction formula.

Theorem 3.2. *The mth power of the generating matrix of generalized Fibonacci sequence of order k, where $m \geq k$, satisfies the following reduction formula:*

$$\mathbf{A}_k^m = \sum_{i=0}^{k-1} \sum_{j=0}^{i} a_{k-i+j} T_{m-j-1} \mathbf{A}_k^i. \tag{7}$$

Proof: Let $m \geq k$ be a positive integer. The product of the characteristic polynomial $p(x) = x^k - a_1 x^{k-1} - \dots - a_k$ and the polynomial $\mathcal{T}(x) = T_{k-1} x^m + T_k x^{m-1} + \dots + T_{m+k-1}$ results in a polynomial with a large number of coefficients that become zero. By multiplying out and combining like terms, the result is expressed as follows:

$$\begin{aligned} \mathcal{T}(x)p(x) = {} & T_{k-1} x^{m+k} + (T_k - a_1 T_{k-1}) x^{m+k-1} + (T_{k+1} - a_1 T_k - a_2 T_{k-1}) x^{m+k-2} \\ & + (T_{k+2} - a_1 T_{k+1} - a_2 T_k - a_3 T_{k-1}) x^{m+k-3} + \cdots \\ & + \left(T_{2k-1} - \sum_{i=1}^{k} a_i T_{2k-i-1} \right) x^m + \dots + \left(T_{m+k-1} - \sum_{i=1}^{k} a_i T_{m+k-i-1} \right) x^k \end{aligned}$$

$$
\begin{aligned}
&- \cdots - (a_{k-3}T_{m+k-1} + a_{k-2}T_{m+k-2} + a_{k-1}T_{m+k-3} + a_k T_{m+k-4})x^3 \\
&- (a_{k-2}T_{m+k-1} + a_{k-1}T_{m+k-2} + a_k T_{m+k-3})x^2 \\
&- (a_{k-1}T_{m+k-1} + a_k T_{m+k-2})x - a_k T_{m+k-1} \\
= {}& T_{k-1}x^{m+k} + \sum_{i=0}^{k-2}\left(T_{k+i} - \sum_{j=0}^{i} a_{j+1}T_{k+i-j-1}\right)x^{m+k-i-1} \\
&+ \left(T_{2k-1} - \sum_{j=0}^{k-1} a_{j+1}T_{2k-j-2}\right)x^m + \cdots + \left(T_{m+k-1} - \sum_{j=0}^{k-1} a_{j+1}T_{m+k-j-2}\right)x^k \\
&- \sum_{i=0}^{k-1}\sum_{j=0}^{i} a_{k-i+j}T_{m+k-j-1}x^i.
\end{aligned}
$$

It is evident that the coefficients of $x^{m+k-1}, \ldots, x^m, \ldots, x^k$ satisfy Equation (1) and thus their values become 0. Applying the Cayley-Hamilton theorem by substituting x with $\mathbf{A}_k$ yields

$$\mathbf{0}_k = \mathcal{T}(\mathbf{A}_k)p(\mathbf{A}_k) = T_{k-1}\mathbf{A}_k^{m+k} - \sum_{i=0}^{k-1}\sum_{j=0}^{i} a_{k-i+j}T_{m+k-j-1}\mathbf{A}_k^i,$$

where $\mathbf{0}_k$ is a square zero matrix of order k. Since $T_{k-1} = 1$, we have

$$\mathbf{A}_k^{m+k} = \sum_{i=0}^{k-1}\sum_{j=0}^{i} a_{k-i+j}T_{m+k-j-1}\mathbf{A}_k^i.$$

We have the desired formula by shifting the power m by $m - k$. □

4. **Applications.** We recall the famous Cassini's identity of the Fibonacci numbers. It can be proved by finding the determinant of $\mathbf{A}_2^n$ where $p = 1$ and $q = -1$:

$$\begin{vmatrix} F_{n+1} & F_{n+2} - F_{n+1} \\ F_n & F_{n+1} - F_n \end{vmatrix} = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^n; \text{ that is, } F_{n-1}F_{n+1} - F_n^2 = (-1)^n.$$

For arbitrary p and q, we have

$$\begin{vmatrix} T_{n+1} & T_{n+1} - pT_{n+1} \\ T_n & T_{n+1} - pT_n \end{vmatrix} = \begin{vmatrix} T_{n+1} & qT_n \\ T_n & qT_{n-1} \end{vmatrix} = \begin{vmatrix} p & q \\ 1 & 0 \end{vmatrix}^n; \text{ that is, } T_{n-1}T_{n+1} - T_n^2 = (-1)^n q^{n-1}.$$

We apply this concept to the determinant of the generating matrix's power for the generalized Fibonacci sequence of order k, thereby deriving the generalized Cassini's identity in the subsequent theorem.

Theorem 4.1. *Let n and k be positive integers. Then*

$$\begin{vmatrix} T_{n+k-1} & T_{n+k} & \cdots & T_{n+2k-2} \\ T_{n+k-2} & T_{n+k-1} & \cdots & T_{n+2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_n & T_{n+1} & \cdots & T_{n+k-1} \end{vmatrix} = \left((-1)^{k+1}a_k\right)^n. \tag{8}$$

Proof: We find the determinant of the power of the generating matrix of the Fibonacci sequence of order k in Equation (6) in two different ways. First, we apply the i-th column operation $C_{i+1} + \sum_{j=1}^{i} a_j C_j$ for $i = 1, 2, \ldots, k-1$ and then find its determinant. This yields

$$\det(\mathbf{A}_k^n) = \begin{vmatrix} T_{n+k-1} & T_{n+k} & \cdots & T_{n+2k-2} \\ T_{n+k-2} & T_{n+k-1} & \cdots & T_{n+2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_n & T_{n+1} & \cdots & T_{n+k-1} \end{vmatrix}.$$

On the other hand, we can find the determinant by diagonalization. Since $\mathbf{A}_k^n = V_k D_k^n V_k^{-1}$ where D_k is a diagonal matrix whose entries on its main diagonal are $\alpha_1, \alpha_2, \ldots, \alpha_k$, we have $\det(\mathbf{A}_k^n) = \big(\det(D_k)\big)^n = (\alpha_1\alpha_2\cdots\alpha_k)^n$. By Viete's formula, the product of the zeros $\alpha_1, \alpha_2, \ldots, \alpha_k$ of the characteristic polynomial in Lemma 2.1 equals $(-1)^{k+1}a_k$. Therefore,

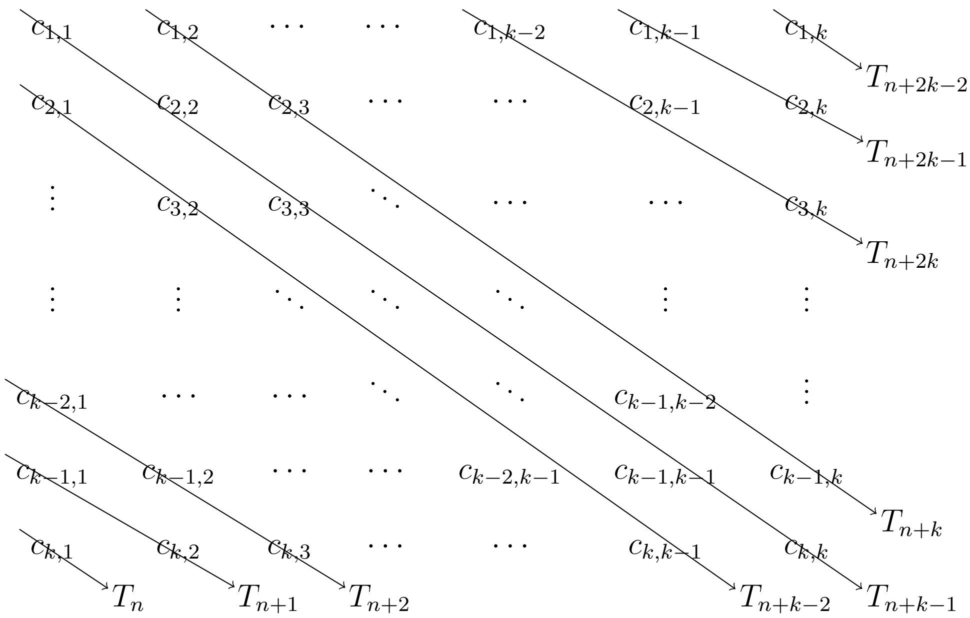
$$\det(\mathbf{A}_k^n) = \Big((-1)^{k+1}a_k\Big)^n$$

and the proof is complete. □

We can see that the matrix in the left-hand side of (8) is a Toeplitz matrix. Since Toeplitz matrix is persymmetric, we can map the entries in the Toeplitz matrix $(c_{i,j})_{1\le i,j\le k}$ as follows:

$$c_{i,j} \mapsto c_{i-j} = T_{n+k-(i-j)-1}.$$

The following diagram shows how this map maps the entries in a Toeplitz matrix of order k.



The determinant of the Toeplitz matrix is explicitly expressed in the following theorem.

Theorem 4.2. *Let n and k be positive integers. Then*

$$\begin{vmatrix} T_{n+k-1} & T_{n+k} & \cdots & T_{n+2k-2} \\ T_{n+k-2} & T_{n+k-1} & \cdots & T_{n+2k-3} \\ \vdots & \vdots & \ddots & \vdots \\ T_n & T_{n+1} & \cdots & T_{n+k-1} \end{vmatrix} = \sum_{t_1+t_2+\cdots+t_k=(k-1)k} \operatorname{sgn}(\sigma) T_{n+t_1}T_{n+t_2}\cdots T_{n+t_k},$$

where $t_\ell = k-(\ell-\sigma(\ell))-1$ for each $\ell = 1, \ldots, k$ and σ is a permutation of $\{1, 2, 3, \ldots, k\}$.

Proof: The formula is precisely the generalized Leibniz's formula for computing the determinant by letting $t_\ell = k - (\ell - \sigma(\ell)) - 1$ for each $\ell = 1, \ldots, k$. Therefore, it suffices to show that

$$\sum_{\ell=1}^{k} t_\ell = \sum_{\ell=1}^{k} \Big(k - \big(\ell - \sigma(\ell)\big) - 1\Big) = (k-1)k.$$

This follows, since $\sum_{\ell=1}^{k} \ell = \sum_{\ell=1}^{k} \sigma(\ell)$. □

Example 4.1. *For $k = 3$, we consider S_3, the permutation group of $\{1, 2, 3\}$. The elements of this group are*

$$\sigma_1 = (\ ), \qquad \sigma_4 = (13),$$
$$\sigma_2 = (123), \qquad \sigma_5 = (23),$$
$$\sigma_3 = (132), \qquad \sigma_6 = (12).$$

For any 3×3 Toeplitz matrix $(c_{i,j})_{i,j}$, we have

$$c_{3,1} \mapsto T_n, \qquad c_{1,3} \mapsto T_{n+4},$$
$$c_{2,1}, c_{3,2} \mapsto T_{n+1}, \qquad c_{1,2}, c_{2,3} \mapsto T_{n+3},$$
$$\text{and} \quad c_{1,1}, c_{2,2}, c_{3,3} \mapsto T_{n+2}.$$

Since σ_1, σ_2, σ_3 are even permutations in S_3, their signs are 1. On the other hand, σ_4, σ_5, σ_6 are odd permutations in S_3, and their signs are -1.

By Theorem 4.2, the determinant of any 3×3 matrix is given by

$$\begin{aligned} & c_{1-\sigma_1(1)}c_{2-\sigma_1(2)}c_{3-\sigma_1(3)} + c_{1-\sigma_2(1)}c_{2-\sigma_2(2)}c_{3-\sigma_2(3)} + c_{1-\sigma_3(1)}c_{2-\sigma_3(2)}c_{3-\sigma_3(3)} \\ & - c_{1-\sigma_4(1)}c_{2-\sigma_4(2)}c_{3-\sigma_4(3)} - c_{1-\sigma_5(1)}c_{2-\sigma_5(2)}c_{3-\sigma_5(3)} - c_{1-\sigma_6(1)}c_{2-\sigma_6(2)}c_{3-\sigma_6(3)} \\ = & T_{n+2}T_{n+2}T_{n+2} + T_{n+3}T_{n+3}T_n + T_{n+4}T_{n+1}T_{n+1} - T_nT_{n+2}T_{n+4} - T_{n+1}T_{n+3}T_{n+2} \\ & - T_{n+2}T_{n+1}T_{n+3}. \end{aligned}$$

Therefore, the determinant of $\mathbf{A}_3^n$ is given by

$$\begin{vmatrix} T_{n+2} & T_{n+3} & T_{n+4} \\ T_{n+1} & T_{n+2} & T_{n+3} \\ T_n & T_{n+1} & T_{n+2} \end{vmatrix} = T_{n+2}^3 + T_nT_{n+3}^2 + T_{n+1}^2T_{n+4} - T_nT_{n+4}T_{n+2} - 2T_{n+1}T_{n+3}T_{n+2}.$$

5. **Conclusions.** In this study, we have demonstrated that the generating matrix for the generalized Fibonacci sequence of order k can be diagonalized by a Vandermonde matrix under specific conditions. This result enables us to derive a closed form for the power of the generating matrix, which in turn allows us to recover well-known identities. Additionally, we provided a closed-form expression for the determinant of a Toeplitz matrix containing generalized Fibonacci sequences.

Acknowledgment. This study was supported by Thammasat University Research Fund, Contract No. TUFT 11/2565. The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

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